Speech Signal Representations

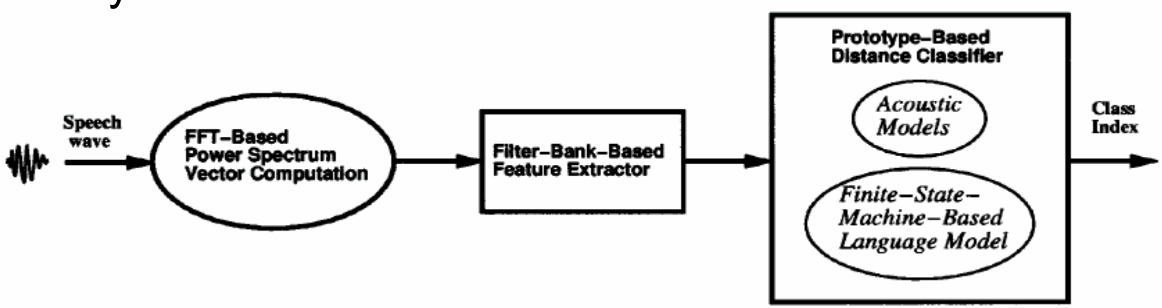
Berlin Chen 2003

References:

1. X. Huang et. al., Spoken Language Processing, Chapters 5, 6
2. J. R. Deller et. al., Discrete-Time Processing of Speech Signals, Chapters 4-6
3. J. W. Picone, “Signal modeling techniques in speech recognition,” proceedings of the IEEE, September 1993, pp. 1215-1247

Introduction

- Current speech recognition systems are mainly composed of：
  - **A front-end feature extractor (feature extraction module)**
    - Required to discover **salient characteristics** suited for classification
    - Based on scientific and/or heuristic knowledge about patterns to recognize
  - **A back-end classifier (classification module)**
    - Required to set class boundaries accurately in the feature space
    - Statistically designed according to the fundamental Bayes' decision theory

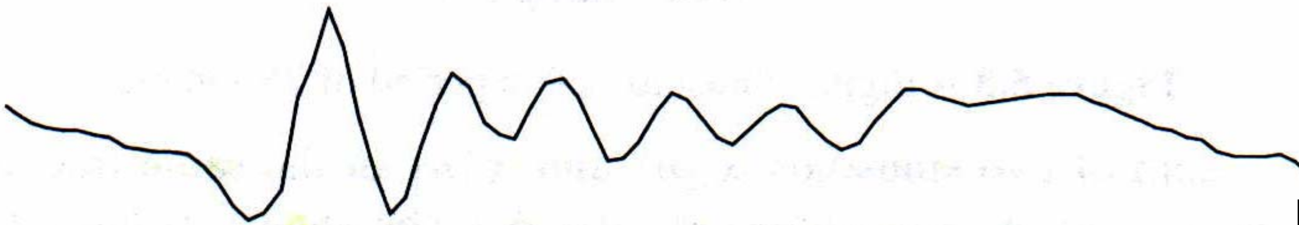


Background Review:

Digital Signal Processing

Analog Signal to Digital Signal

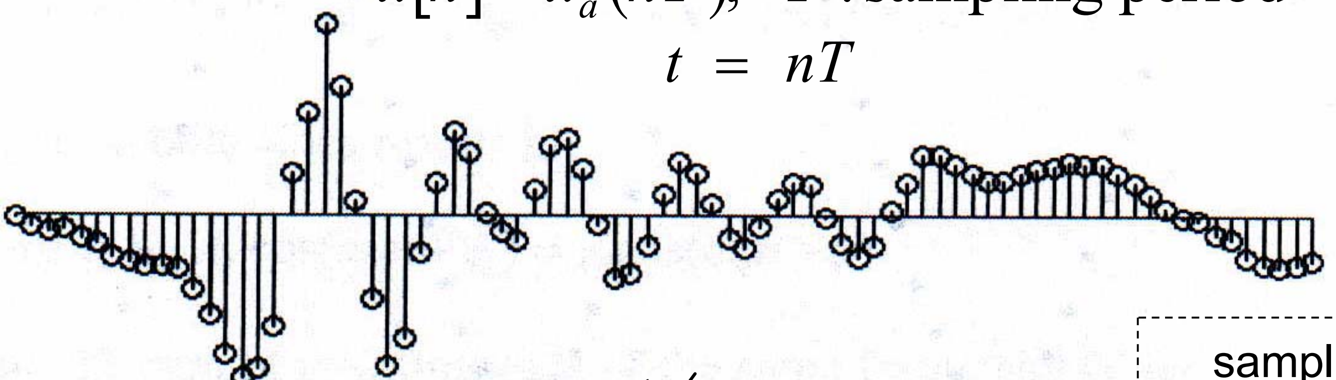
Analog Signal



Discrete-time Signal or Digital Signal

$$x[n] = x_a(nT), \quad T: \text{sampling period}$$

$$t = nT$$

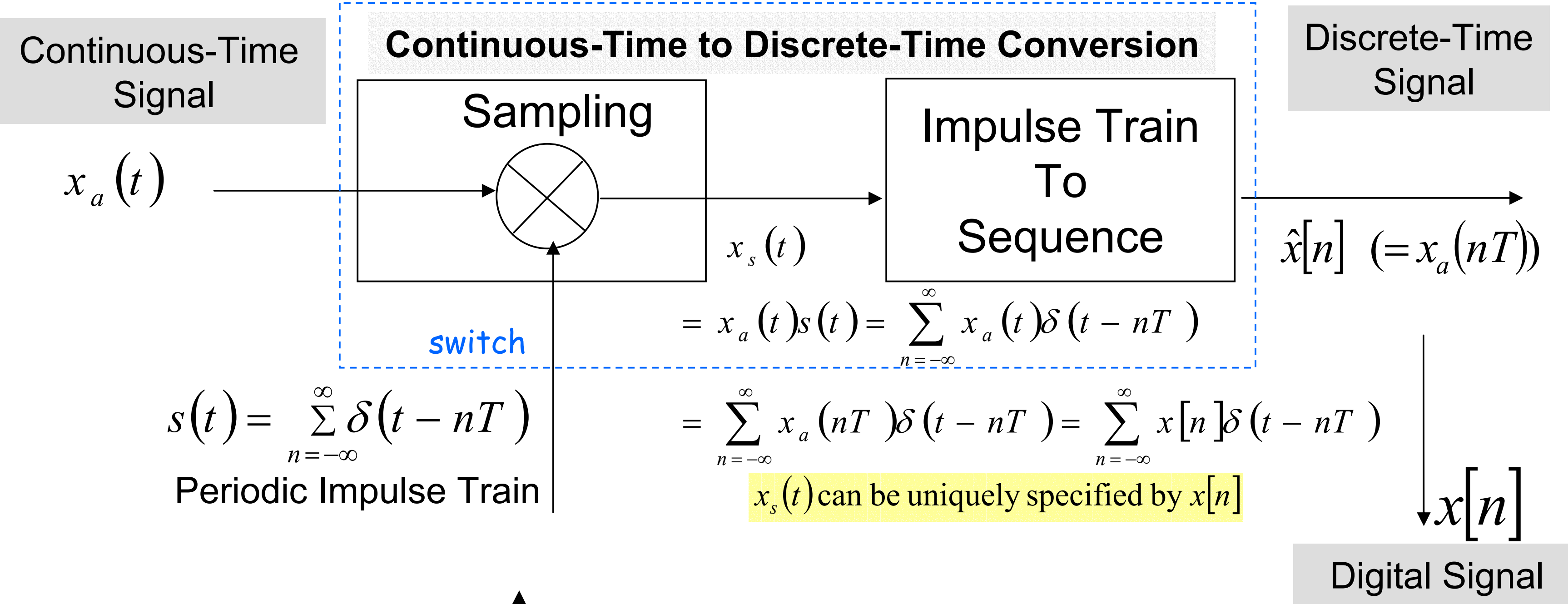


$$F_s = 1/T \quad \text{sampling rate}$$

Digital Signal:
Discrete-time signal with discrete amplitude

sampling period=125 μs
=>sampling rate=8kHz

Analog Signal to Digital Signal



$x_a(t)$ → Sampling → $x_s(t)$ → Impulse Train To Sequence → $\hat{x}[n] \; (= x_a(nT))$ → $x[n]$

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

Periodic Impulse Train

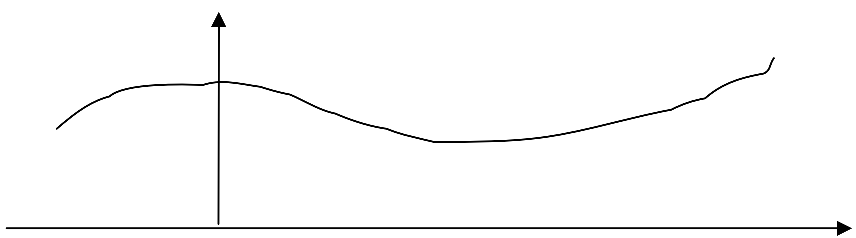
$$x_s(t) = x_a(t)s(t) = \sum_{n=-\infty}^{\infty} x_a(t)\delta(t - nT)$$

$$= \sum_{n=-\infty}^{\infty} x_a(nT)\delta(t - nT) = \sum_{n=-\infty}^{\infty} x[n]\delta(t - nT)$$

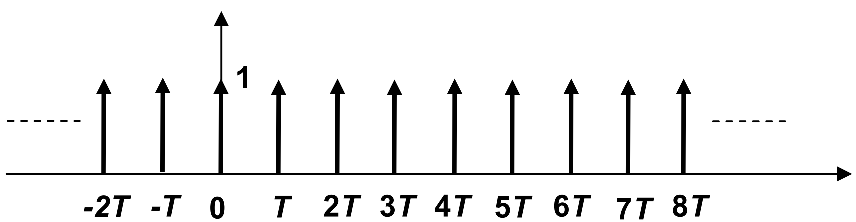
$x_s(t)$ can be uniquely specified by $x[n]$

Discrete-time signal with discrete amplitude

$x_a(t)$



$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

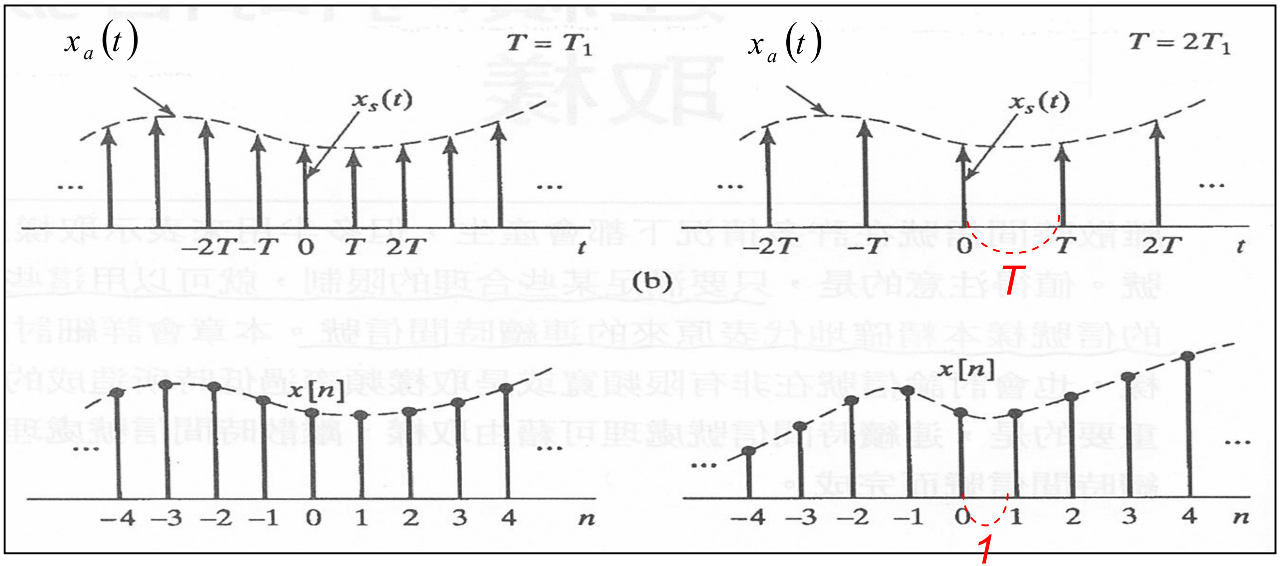


$$\delta(t) = 0, \quad \forall t \neq 0$$

$$\int_{-\infty}^{\infty} \delta(t)dt = 1$$

Analog Signal to Digital Signal

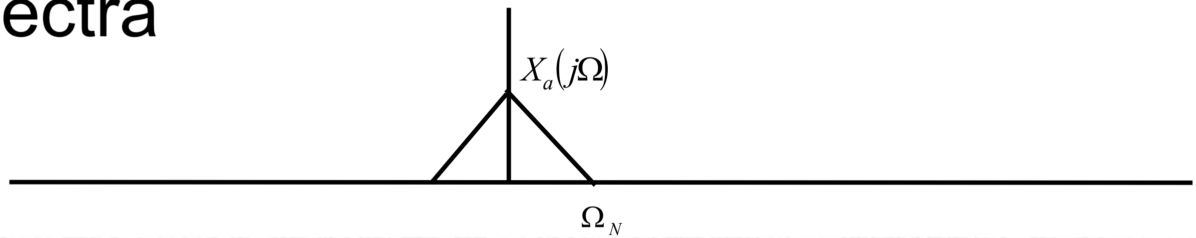
- A continuous signal sampled at different periods



$$
\begin{aligned}
x_s(t) \\
&= x_a(t)s(t) = \sum_{n=-\infty}^{\infty} x_a(t)\delta(t-nT) \\
&= \sum_{n=-\infty}^{\infty} x_a(nT)\delta(t-nT) = \sum_{n=-\infty}^{\infty} x[n]\delta(t-nT)
\end{aligned}
$$

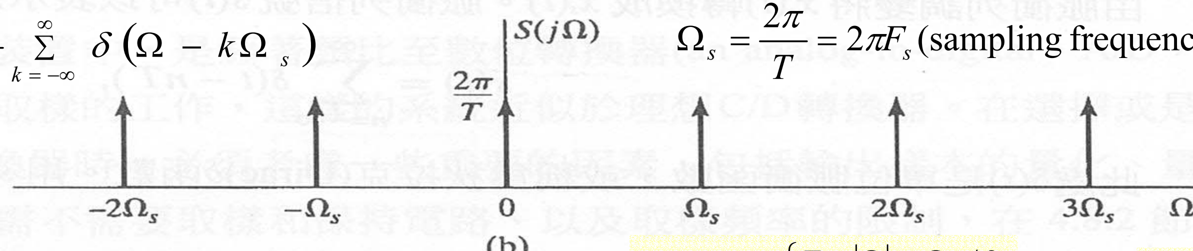
Analog Signal to Digital Signal

- Spectra



$$S(j\Omega) = \frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta(\Omega - k\Omega_s)$$

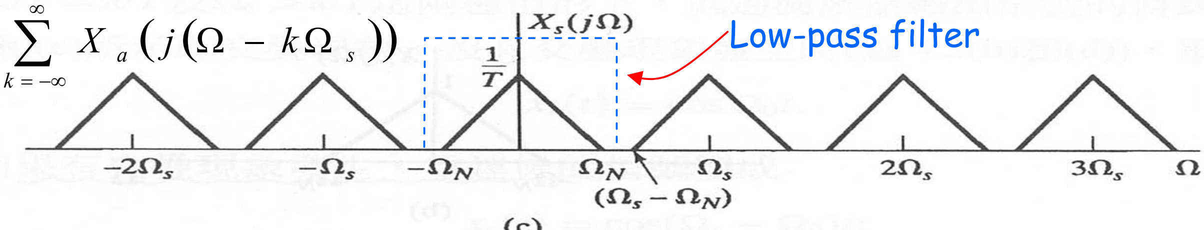
$$\Omega_s = \frac{2\pi}{T} = 2\pi F_s \text{ (sampling frequency)}$$



$$X_s(j\Omega) = \frac{1}{2\pi}X_a(j\Omega) * S(j\Omega)$$

$$R_{\Omega_s}(j\Omega) = \begin{cases} T & |\Omega| < \Omega_s/2 \\ 0 & \text{otherwise} \end{cases} \Rightarrow X_a(j\Omega) = R_{\Omega_s}(j\Omega)X_p(j\Omega)$$

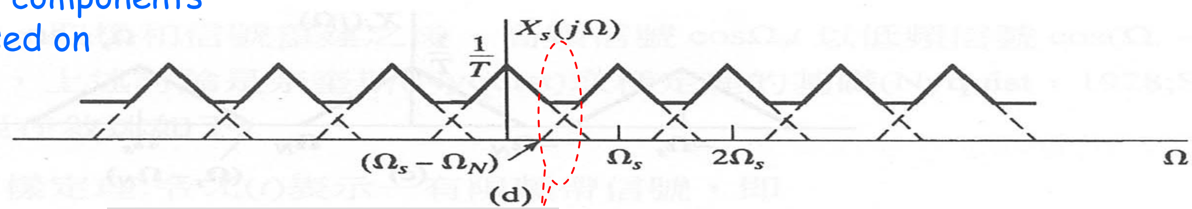
$$X_s(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty}X_a(j(\Omega - k\Omega_s))$$



$$\Omega_N < \frac{1}{2}\Omega_s\left(=\frac{\pi}{T}\right)$$

$$\left\{\begin{array}{l} \because (\Omega_s - \Omega_N) > \Omega_N \\ \Rightarrow \Omega_s > 2\Omega_N \end{array}\right\}$$

high frequency components got superimposed on low frequency components



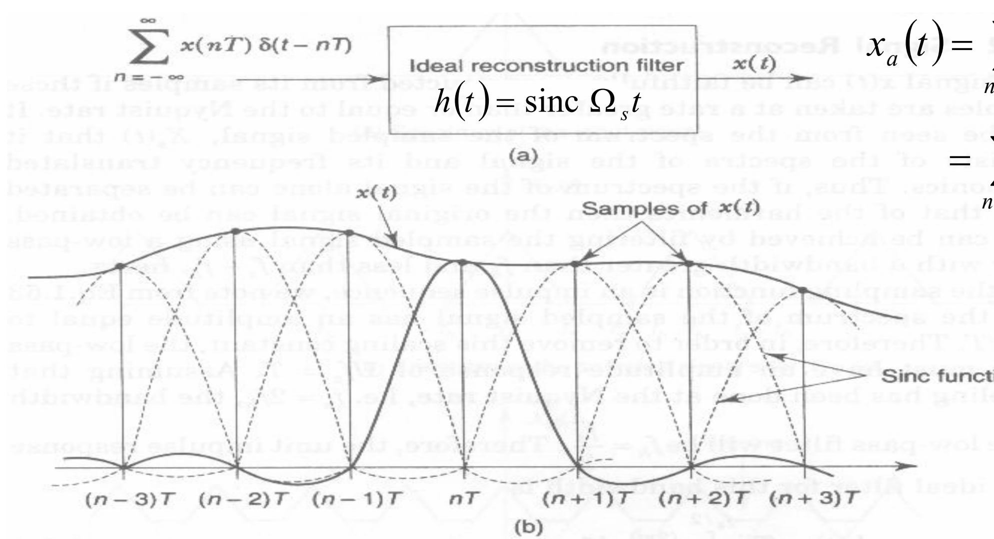
$$\Omega_N > \frac{1}{2}\Omega_s\left(=\frac{\pi}{T}\right)$$

$$\left\{\begin{array}{l} \because (\Omega_s - \Omega_N) < \Omega_N \\ \Rightarrow \Omega_s < 2\Omega_N \end{array}\right\}$$

aliasing distortion ⇒ $X_a(j\Omega)$ can't be recovered from $X_p(j\Omega)$

Analog Signal to Digital Signal

- To avoid aliasing (*overlapping*, *fold over*)
  - The sampling frequency should be greater than two times of frequency of the signal to be sampled → $\Omega_s > 2\Omega_N$
  - (Nyquist) sampling theorem
- To reconstruct the original continuous signal
  - Filtered with a low pass filter with band limit Ω_s
    - Convolved in time domain

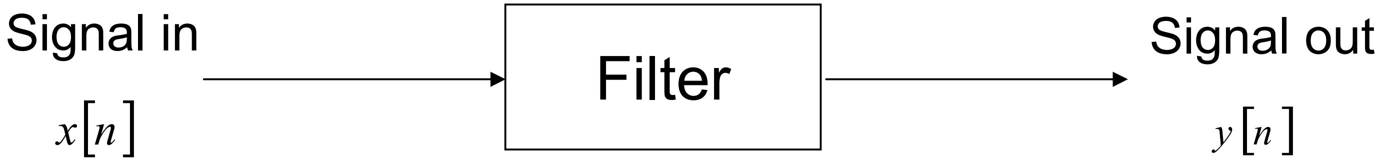


$$x_a(t) = \sum_{n=-\infty}^{\infty} x_a(nT)\,h(t-nT)$$

$$= \sum_{n=-\infty}^{\infty} x_a(nT)\,\text{sinc}\,\Omega_s(t-nT)$$

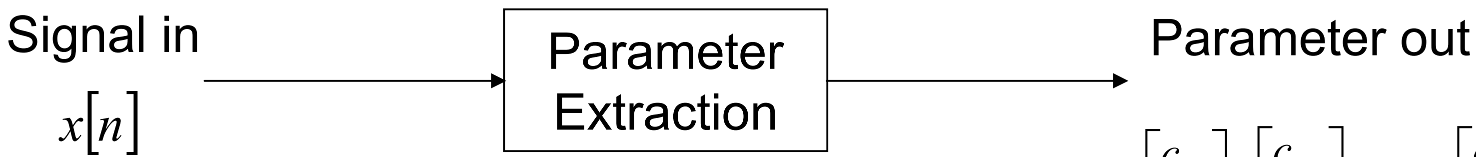
Two Main Approaches to Digital Signal Processing

- Filtering



Amplify or attenuate some frequency components of $x[n]$

- Parameter Extraction



$$\begin{bmatrix} c_{11} \\ c_{12} \\ \vdots \\ c_{1m} \end{bmatrix} \begin{bmatrix} c_{21} \\ c_{22} \\ \vdots \\ c_{2m} \end{bmatrix} \cdots \begin{bmatrix} c_{L1} \\ c_{L2} \\ \vdots \\ c_{Lm} \end{bmatrix}$$

e.g.:

1. Spectrum Estimation
2. Parameters for Recognition

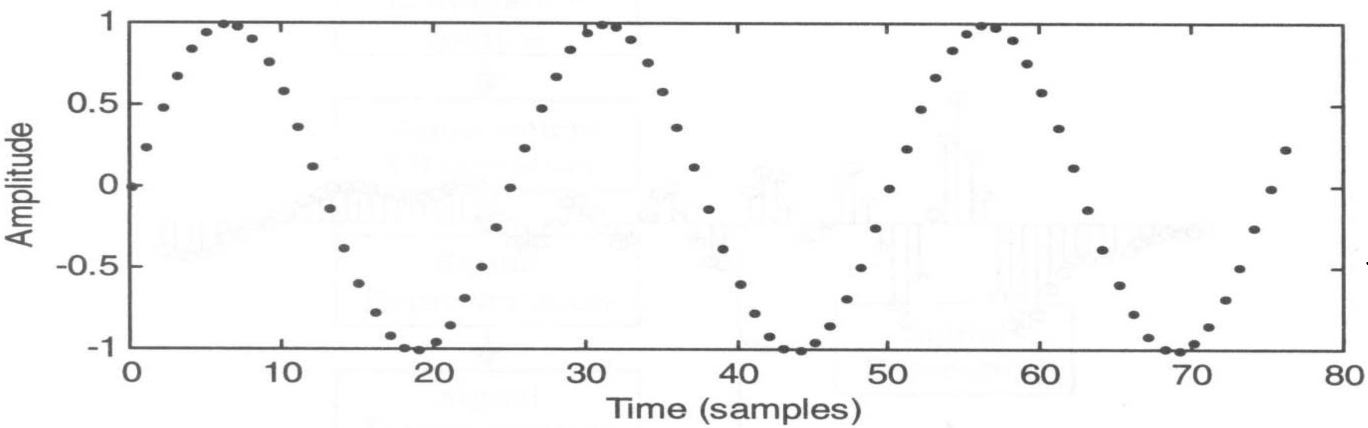
Sinusoid Signals

$$x[n] = A\cos(\omega n + \phi)$$

f : normalized frequency
$0 \le f \le 1$

- A : amplitude (振幅)
- ω : angular frequency (角頻率), $\omega = 2\pi f = \frac{2\pi}{T}$
- ϕ : phase (相角)

Period, represented by number of samples



$$x[n] = A\cos\left(\omega n - \frac{\pi}{2}\right)$$

$T = 25$ samples

Sinusoid Signals

- $x[n]$ is periodic with a period of *N* (samples)

$$\Rightarrow \quad x[n+N] = x[n]$$

$$\Rightarrow \quad A\cos(\omega(n+N)+\phi) = A\cos(\omega n+\phi)$$

$$\Rightarrow \quad \omega N = 2\pi$$

$$\Rightarrow \quad \omega = \frac{2\pi}{N}$$

- Examples (sinusoid signals)

  – $x_1[n] = \cos(\pi n/4)$ is periodic with period *N*=8
  – $x_2[n] = \cos(3\pi n/8)$ is periodic with period *N*=16
  – $x_3[n] = \cos(n)$ is not periodic

Sinusoid Signals

$$x_1[n] = \cos(\pi n / 4)$$

$$= \cos\left(\frac{\pi}{4} n\right) = \cos\left(\frac{\pi}{4}(n + N_1)\right) = \cos\left(\frac{\pi}{4} n + \frac{\pi}{4} N_1\right)$$

$$\Rightarrow \frac{\pi}{4} N_1 = 2\pi \cdot k \Rightarrow 8 \cdot k \qquad (N_1 \text{ and } k \text{ are positive integers})$$

$$\therefore N_1 = 8$$

$$x_2[n] = \cos(3\pi n / 8)$$

$$= \cos\left(\frac{3\pi}{8} \cdot n\right) = \cos\left(\frac{3\pi}{8}(n + N_2)\right) = \cos\left(\frac{3\pi}{8} \cdot n + \frac{3\pi}{8} \cdot N_2\right)$$

$$\Rightarrow \frac{3\pi}{8} \cdot N_2 = 2\pi \cdot k \Rightarrow N_2 = \frac{16}{3} k \qquad (N_2 \text{ and } k \text{ are positive numbers})$$

$$\therefore N_2 = 16$$

$$x_3[n] = \cos(n)$$

$$= \cos(1 \cdot n) = \cos(1 \cdot (n + N_3)) = \cos(n + N_3)$$

$$\Rightarrow N_3 = 2\pi \cdot k$$

$$\because N_3 \text{ and } k \text{ are positive integers}$$

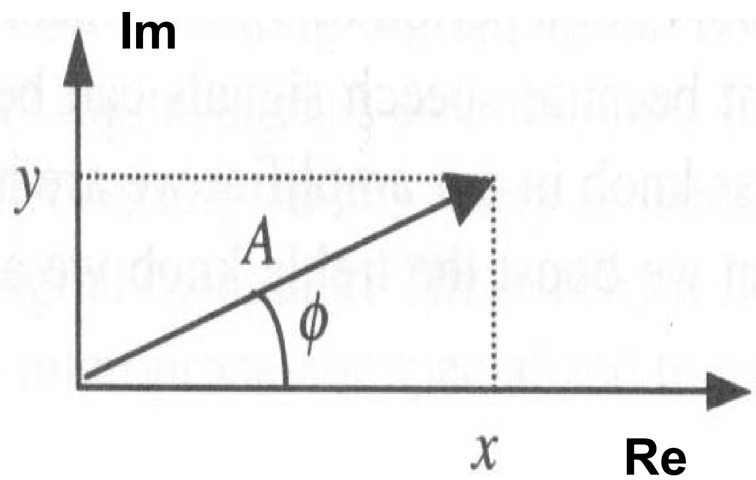
$$\therefore N_3 \text{ doesn't exist !}$$

Sinusoid Signals

- Complex Exponential Signal
  - Use **Euler's relation** to express complex numbers

$$z = x + jy$$

$$\Rightarrow z = Ae^{j\phi} = A(\cos\phi + j\sin\phi) \quad (A \text{ is a real number})$$



$$x = A\cos\phi$$

$$y = A\sin\phi$$

Sinusoid Signals

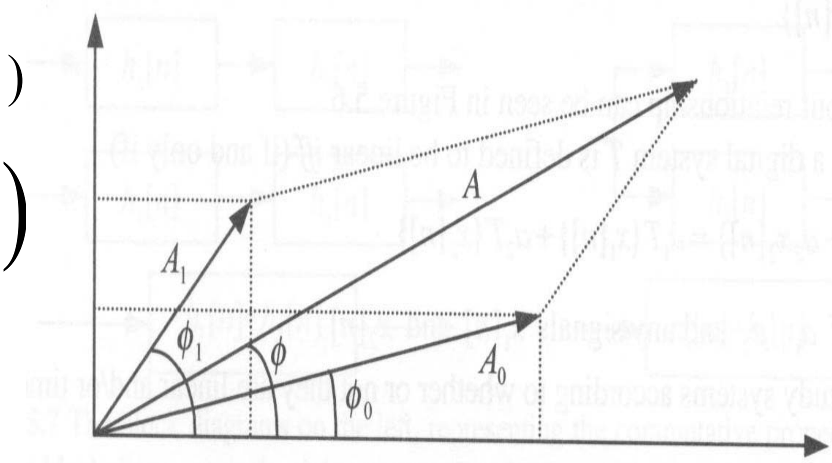
- A Sinusoid Signal

$$x[n] = A\cos(\omega n + \phi)$$
$$= \mathrm{Re}\left\{Ae^{j(\omega n + \phi)}\right\}$$
$$= \mathrm{Re}\left\{Ae^{j\omega n}e^{j\phi}\right\}$$

Sinusoid Signals

- Sum of two complex exponential signals with same frequency

$$A_0 e^{j(\omega n+\phi_0)} + A_1 e^{j(\omega n+\phi_1)}$$
$$= e^{j\omega n}\left(A_0 e^{j\phi_0} + A_1 e^{j\phi_1}\right)$$
$$= e^{j\omega n} A e^{j\phi}$$
$$= Ae^{j(\omega n+\phi)}$$



A, A_0 and A_1 are real numbers

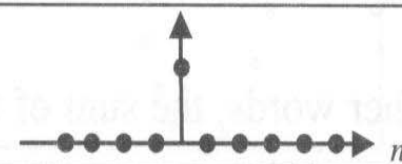
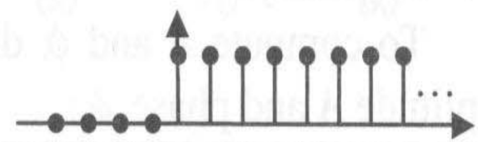
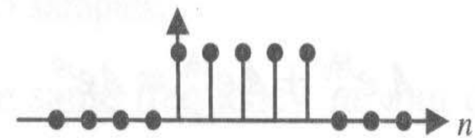
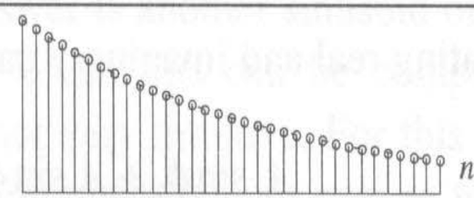
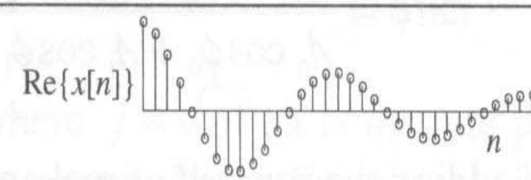
- When only the real part is considered

$$A_0 \cos(\omega n + \phi_0) + A_1 \cos(\omega n + \phi_1) = A\cos(\omega n + \phi)$$

- The sum of *N* sinusoids of the same frequency is another sinusoid of the same frequency

Some Digital Signals

Table 5.1 Some useful digital signals: the Kronecker delta, unit step, rectangular signal, real exponential ($a < 1$) and real part of a complex exponential ($r < 1$).

Kronecker delta, or *unit impulse*	$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases}$	
Unit step	$u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$	
Rectangular signal	$\text{rect}_N[n] = \begin{cases} 1 & 0 \leq n < N \\ 0 & \text{otherwise} \end{cases}$	
Real exponential	$x[n] = a^n u[n]$	
Complex exponential	$x[n] = a^n u[n] = r^n e^{jn\omega_0} u[n]$ $= r^n (\cos n\omega_0 + j \sin n\omega_0) u[n]$	

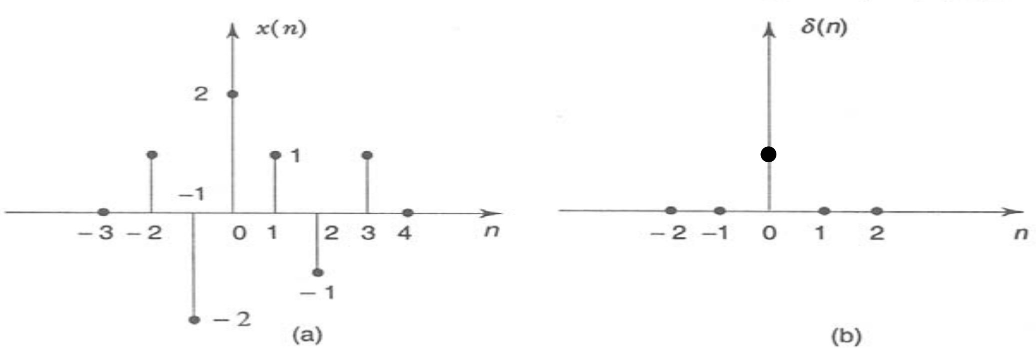
Some Digital Signals

- Any signal sequence $x[n]$ can be represented as a sum of *shift* and *scaled* unit impulse sequences (signals)

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$$

scale/weighted

Time-shifted unit impulse sequence



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$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k] = \sum_{k=-2}^{3} x[k]\,\delta[n-k]$$

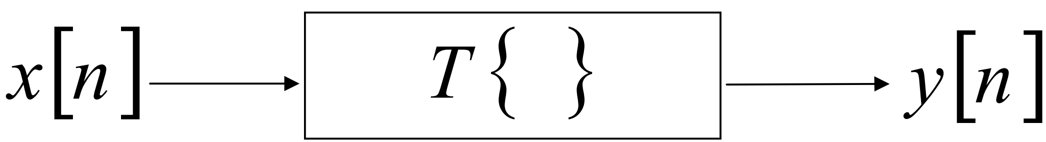
$$= x[-2]\delta[n+2] + x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1] + x[2]\delta[n-2] + x[3]\delta[n-3]$$

$$= (1)\delta[n+2] + (-2)\delta[n+1] + (2)\delta[n] + (1)\delta[n-1] + (-1)\delta[n-2] + (1)\delta[n-3]$$

Digital Systems

- A digital system T is a system that, given an input signal $x[n]$, generates an output signal $y[n]$

$$y[n] = T\{x[n]\}$$



Properties of Digital Systems

- Linear
  - Linear combination of inputs maps to linear combination of outputs

$$T\{ax_1[n]+bx_2[n]\}=aT\{x_1[n]\}+bT\{x_2[n]\}$$

- Time-invariant (Time-shift)
  - A time shift of in the input by *m* samples give a shift in the output by *m* samples

$$y[n\pm m]=T\{x[n\pm m]\},\quad \forall m$$

Properties of Digital Systems

- Linear time-invariant (LTI)
  - The system output can be expressed as a convolution (迴旋積分) of the input $x[n]$ and the *impulse response* $h[n]$
  - The system can be characterized by the system's *impulse response* $h[n]$, which also is a signal sequence
    - If the input $x[n]$ is impulse $\delta[n]$, the output is $h[n]$

Properties of Digital Systems

- Linear time-invariant (LTI)
  - Explanation:

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$$

scale: $x[k]$

Time-shifted unit impulse sequence: $\delta[n-k]$

$$\Rightarrow T\{x[n]\} = T\left\{\sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]\right\}$$

linear

$$= \sum_{k=-\infty}^{\infty} x[k]\,T\{\delta[n-k]\}$$

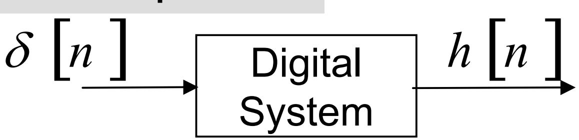
Time-invariant

$$= \sum_{k=-\infty}^{\infty} x[k]\,h[n-k]$$

$$= x[n] * h[n]$$

convolution

Impulse response



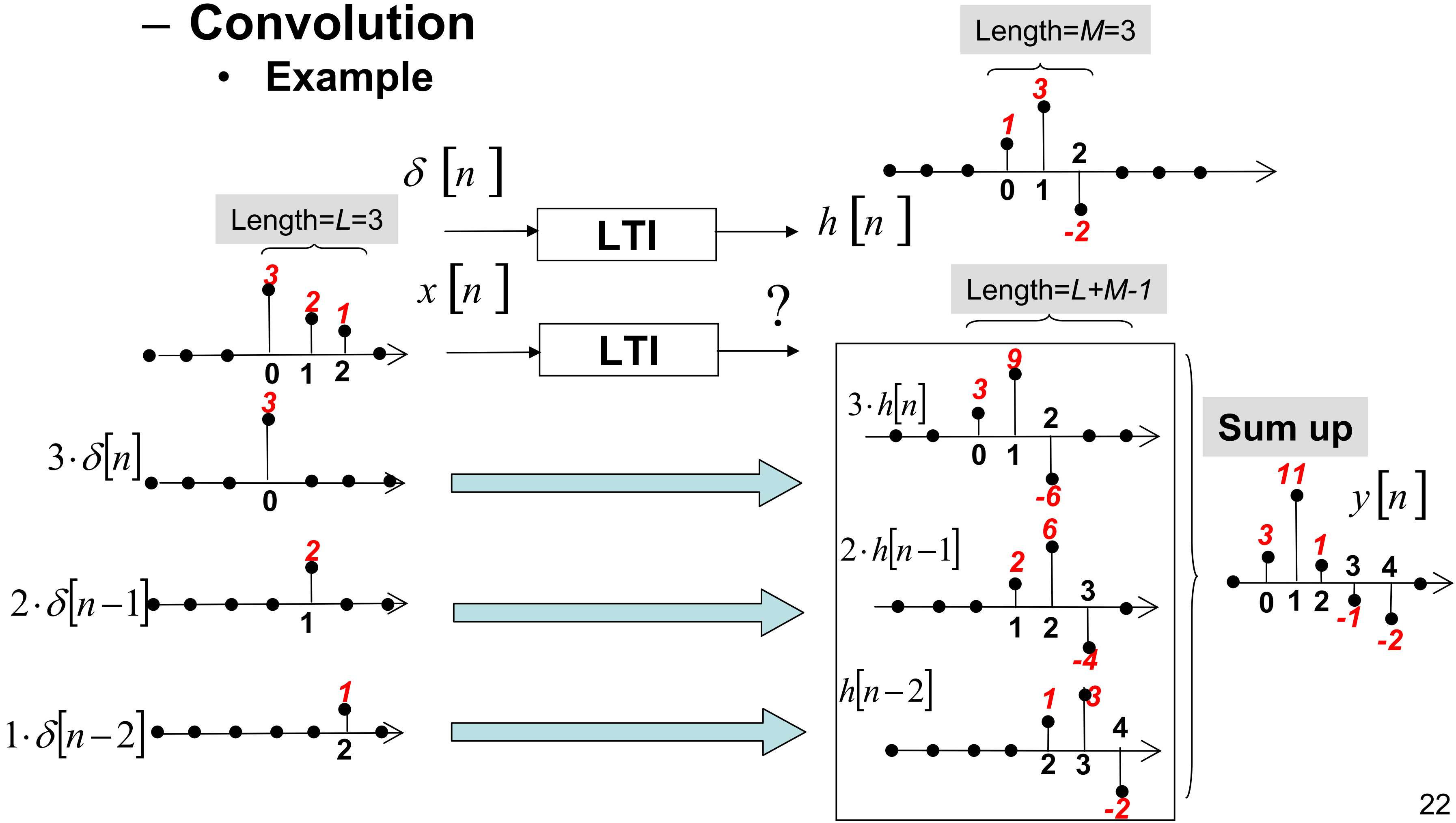
Time invariant

$$\delta[n] \xrightarrow{T} h[n]$$

$$\delta[n-k] \xrightarrow{T} h[n-k]$$

Properties of Digital Systems

- Linear time-invariant (LTI)
  - **Convolution**
    - **Example**

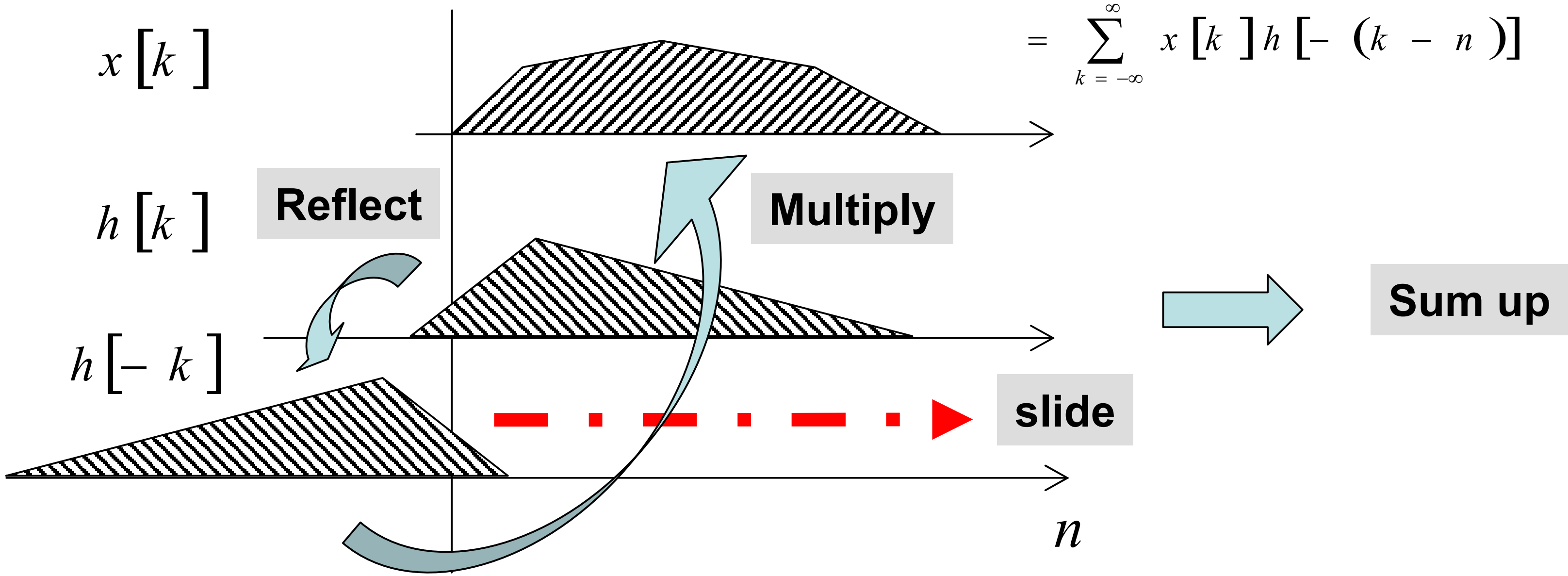


Properties of Digital Systems

- Linear time-invariant (LTI)
  - Convolution: **Generalization**
    - Reflect $h[k]$ about the origin ($\rightarrow h[-k]$)
    - Slide ($h[-k] \rightarrow h[-k+n]$ or $h[-(k-n)]$), multiply it with $x[k]$
    - Sum up

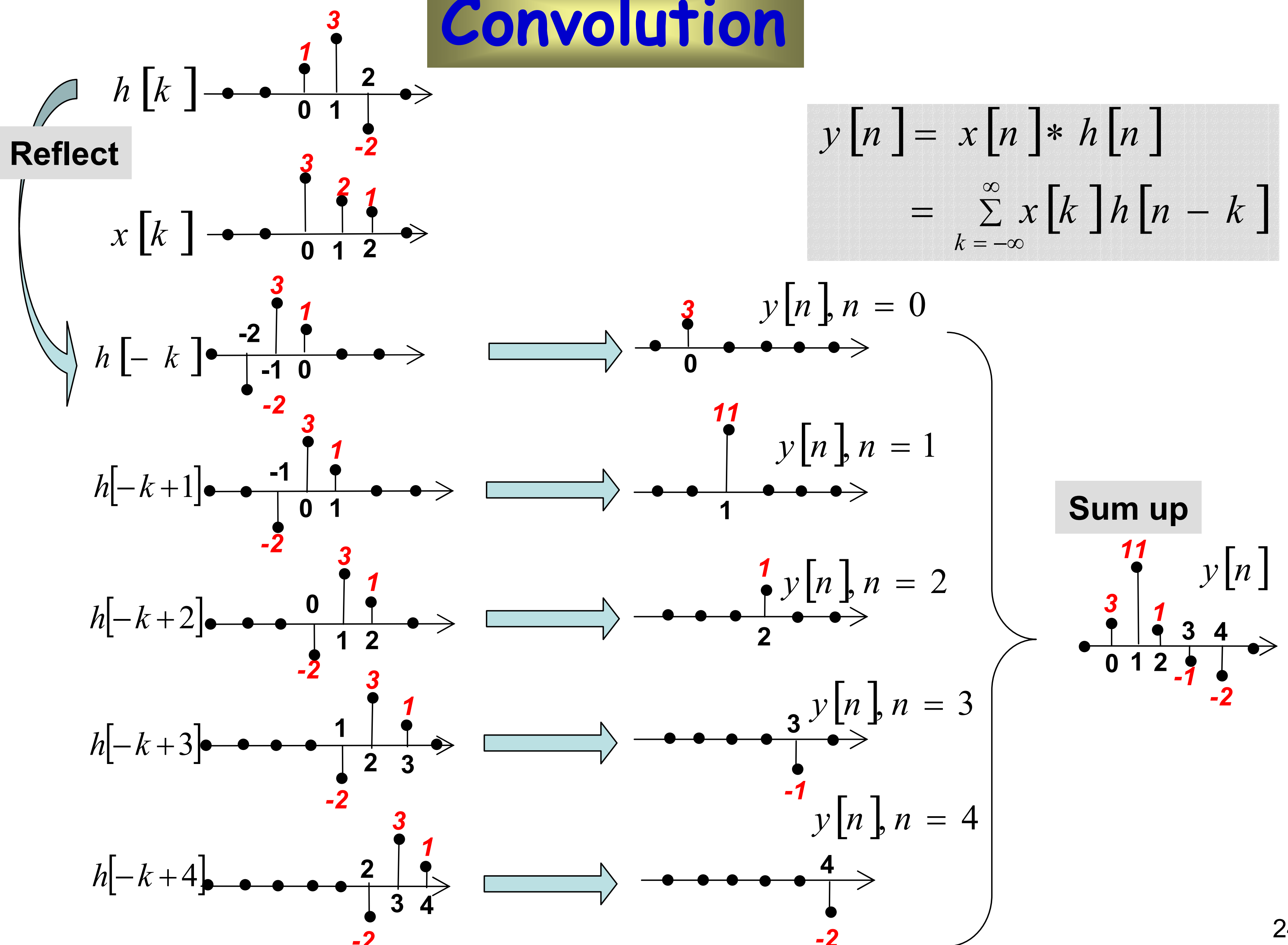
$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

$$= \sum_{k=-\infty}^{\infty} x[k]h[-(k-n)]$$



Convolution

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k]$$



Properties of Digital Systems

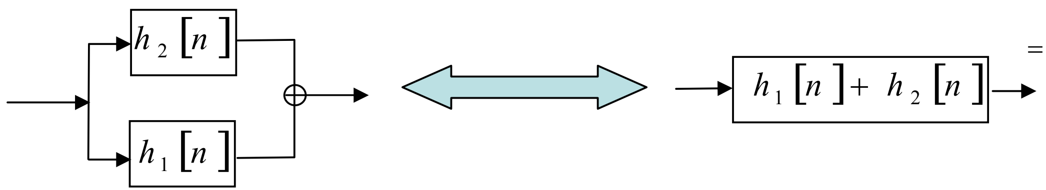
- Linear time-invariant (LTI)
  - Convolution is commutative and distributive

$$x[n]*h_1[n]*h_2[n] = x[n]*h_2[n]*h_1[n]$$

Commutation

$$x[n]*(h_1[n]+h_2[n]) = x[n]*h_1[n]+x[n]*h_2[n]$$

Distribution



– An impulse response has finite duration

» **Finite-Impulse Response (FIR)**

– An impulse response has infinite duration

» **Infinite-Impulse Response (IIR)**

$$\begin{aligned} y[n] &= x[n]*h[n] \\ &= h[n]*x[n] \\ &= \sum_{k=-\infty}^{\infty} x[k]h[n-k] \\ &= \sum_{k=-\infty}^{\infty} h[k]x[n-k] \end{aligned}$$

Properties of Digital Systems

- **Bounded Input and Bounded Output (BIBO)**: stable

$$|x[n]| \leq B_x < \infty \quad \forall n$$
$$|y[n]| \leq B_y < \infty \quad \forall n$$

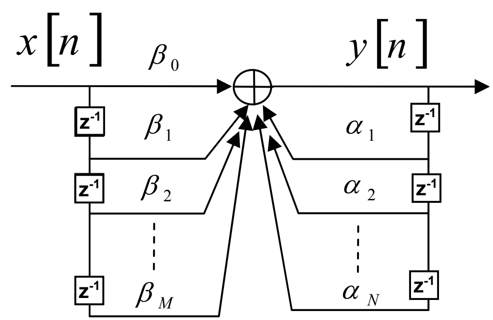
  - **A LTI system** is BIBO if only if *h*[*n*] is absolutely summable

$$\sum_{k=-\infty}^{\infty} |h[k]| \leq \infty$$

Properties of Digital Systems

- **Causality**
  - A system is “casual” if for every choice of n_0, the output sequence value at indexing $n=n_0$ depends on only the input sequence value for $n \leq n_0$

$$y[n_0] = \sum_{k=1}^{K} \alpha_k y[n_0 - k] + \sum_{k=m}^{M} \beta_k x[n_0 - m]$$



  - Any noncausal FIR can be made causal by adding sufficient long delay

Discrete-Time Fourier Transform (DTFT)

- Frequency Response $H\left(e^{j\omega}\right)$
  - Defined as the discrete-time Fourier Transform of $h[n]$
  - $H\left(e^{j\omega}\right)$ is continuous and is periodic with period= 2π

proportional to two times of the sampling frequency

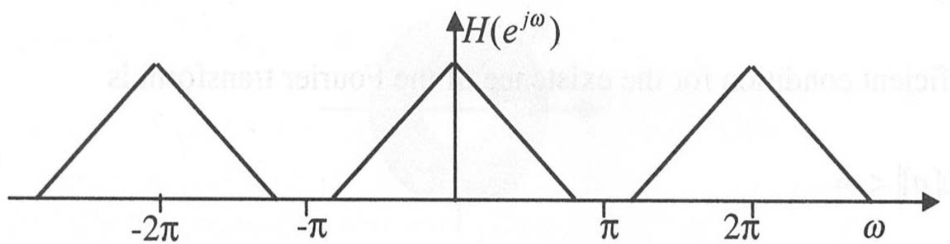


Figure 5.8 $H(e^{j\omega})$ is a periodic function of ω.

  - $H\left(e^{j\omega}\right)$ is a complex function of ω

$$H\left(e^{j\omega}\right) = H_r\left(e^{j\omega}\right) + jH_i\left(e^{j\omega}\right)$$

$$= \left|H\left(e^{j\omega}\right)\right| e^{j\angle H\left(e^{j\omega}\right)}$$

magnitude

phase

Discrete-Time Fourier Transform

- Representation of Sequences by Fourier Transform

$$H\left(e^{j\omega}\right)=\sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n}$$ DTFT

$$h[n]=\frac{1}{2\pi}\int_{-\pi}^{\pi} H\left(e^{j\omega}\right)e^{j\omega n}\,d\omega$$ Inverse DTFT

  - A sufficient condition for the existence of Fourier transform

$$\sum_{n=-\infty}^{\infty} \left|h[n]\right| < \infty$$ *absolutely summable*

Fourier transform is invertible:

$$H\left(e^{j\omega}\right)=\sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n}$$

$$h[n]=\frac{1}{2\pi}\int_{-\pi}^{\pi} H\left(e^{j\omega}\right)e^{j\omega n}d\omega=\frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{m=-\infty}^{\infty} h[m]e^{-j\omega m}e^{j\omega n}d\omega$$

$$=\sum_{m=-\infty}^{\infty} h[m]\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\omega(n-m)}d\omega=\sum_{m=-\infty}^{\infty} h[m]\delta[n-m]=h[n]$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\omega(n-m)}d\omega$$

$$=\frac{1}{j2\pi(n-m)}\left[e^{j\omega(n-m)}\right]_{-\pi}^{\pi}$$

$$=\frac{\sin\pi(n-m)}{\pi(n-m)}$$

$$=\begin{cases}1, & m=n\\ 0, & m\neq n\end{cases}$$

$$=\delta[n-m]$$

Discrete-Time Fourier Transform

- Convolution Property

$$H\left(e^{j\omega}\right)=\sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n}$$

$$y[n] = x[n]*h[n]=\sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

$$Y\left(e^{j\omega}\right)=\sum_{n=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} x[k]h[n-k]e^{-j\omega n}$$

$$n' = n-k$$
$$\Rightarrow n = n'+k$$
$$\Rightarrow -n = -n'-k$$

$$=\sum_{k=-\infty}^{\infty} x[k]e^{-j\omega k}\left(\sum_{n'=-\infty}^{\infty} h[n']e^{-j\omega n'}\right)$$

$$= X\left(e^{j\omega}\right)H\left(e^{j\omega}\right)$$

$$\therefore\ x[n]*h[n]\Leftrightarrow X\left(e^{j\omega}\right)H\left(e^{j\omega}\right)$$

$$Y\left(e^{j\omega}\right)= X\left(e^{j\omega}\right)H\left(e^{j\omega}\right)$$
$$\Rightarrow \left|Y\left(e^{j\omega}\right)\right| = \left|X\left(e^{j\omega}\right)\right|\left|H\left(e^{j\omega}\right)\right|$$
$$\Rightarrow \angle Y\left(e^{j\omega}\right)= \angle X\left(e^{j\omega}\right)+\angle H\left(e^{j\omega}\right)$$

Discrete-Time Fourier Transform

- Parseval's Theorem

power spectrum

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|X\left(e^{j\omega}\right)\right|^2 d\omega$$

The total energy of a signal can be given in either the time or frequency domain.

– Define the autocorrelation of signal $x[n]$

$$R_{xx}[n] = \sum_{m=-\infty}^{\infty} x[m+n]x^*[m]$$

$$l = m + n \Rightarrow m = l - n = -(n-l)$$

$$\Leftrightarrow \quad = \sum_{l=-\infty}^{\infty} x[l]x^*[-(n-l)] = x[n] * x^*[-n]$$

$$S_{xx}(\omega) = X(\omega)X^*(\omega) = |X(\omega)|^2$$

$$R_{xx}[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} S_{xx}(\omega)e^{j\omega n}d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(\omega)|^2 e^{j\omega n}d\omega$$

Set $n = 0$

$$R_{xx}[0] = \sum_{m=-\infty}^{\infty} x[m]x^*[m] = \sum_{m=-\infty}^{\infty} |x[m]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(\omega)|^2 d\omega$$

Discrete-Time Fourier Transform

Property	Signal	Fourier Transform
Linearity	$ax_1[n]+bx_2[n]$	$aX_1(e^{j\omega})+bX_2(e^{j\omega})$
Symmetry	$x[-n]$	$X(e^{-j\omega})$
	$x^*[n]$	$X^*(e^{-j\omega})$
	$x^*[-n]$	$X^*(e^{j\omega})$
	$x[n]$ real	$X(e^{j\omega})$ is *Hermitian* $X(e^{-j\omega})=X^*(e^{j\omega})$ $\lvert X(e^{j\omega})\rvert$ is even[6] $\mathrm{Re}\{X(e^{j\omega})\}$ is even $\arg\{X(e^{j\omega})\}$ is odd[7] $\mathrm{Im}\{X(e^{j\omega})\}$ is odd
	$\mathrm{Even}\{x[n]\}$	$\mathrm{Re}\{X(e^{j\omega})\}$
	$\mathrm{Odd}\{x[n]\}$	$j\,\mathrm{Im}\{X(e^{j\omega})\}$
Time-shifting	$x[n-n_0]$	$X(e^{j\omega})e^{-j\omega n_0}$
Modulation	$x[n]e^{j\omega_0 n}$	$X(e^{j(\omega-\omega_0)})$
	$x[n]z_0^n$	
Convolution	$x[n]*h[n]$	$X(e^{j\omega})H(e^{j\omega})$
	$x[n]y[n]$	$\frac{1}{2\pi}X(e^{j\omega})*Y(e^{j\omega})$
Parseval's Theorem	$R_{xx}[n]=\sum_{m=-\infty}^{\infty}x[m+n]x^*[m]$	$S_{xx}(\omega)=\lvert X(\omega)\rvert^2$

Z-Transform

- *z*-transform is a generalization of (Discrete-Time) Fourier transform

$$h[n] \longrightarrow H(e^{j\omega})$$

$$h[n] \longrightarrow H(z)$$

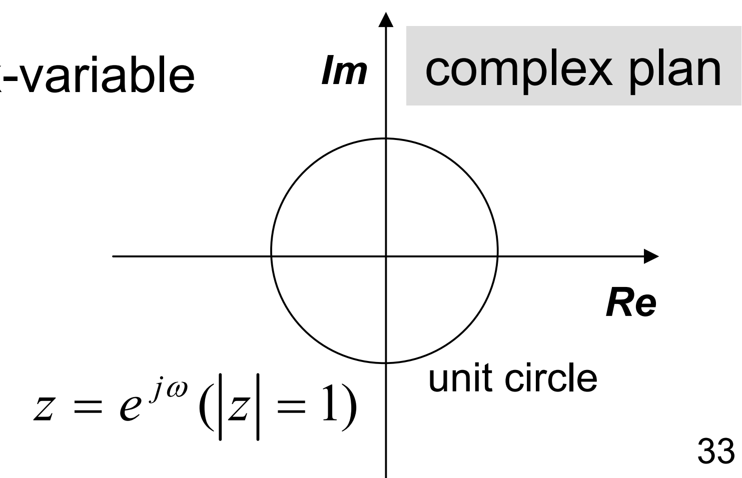
  – z-transform of $h[n]$ is defined as

$$H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n}$$

  - Where $z = re^{j\omega}$, a complex-variable
  - For Fourier transform

$$H(e^{j\omega}) = H(z)\Big|_{z=e^{j\omega}}$$

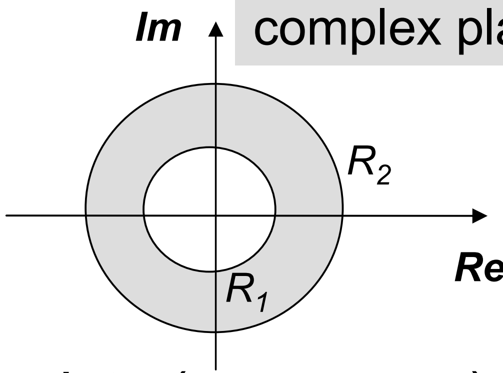
    – z-transform evaluated on the unit circle



Z-Transform

- Fourier transform vs. *z*-transform
  - Fourier transform used to plot the frequency response of a filter
  - z-transform used to analyze more general filter characteristics, e.g. stability

complex plan



- ROC (Region of Converge)
  - Is the set of *z* for which *z*-transform exists (converges)

$$\sum_{n=-\infty}^{\infty} \left|h[n]\right| \left|z\right|^{-n} < \infty$$

absolutely summable

  - In general, ROC is a **ring-shaped region** and the Fourier transform exists if ROC includes the unit circle

Z-Transform

$$\begin{aligned} y[n] &= x[n] * h[n] \\ &= h[n] * x[n] \\ &= \sum_{k=-\infty}^{\infty} x[k]\, h[n-k] \\ &= \sum_{k=-\infty}^{\infty} h[k]\, x[n-k] \end{aligned}$$

- An LTI system is defined to be ***causal***, if its impulse response is a causal signal, i.e.

$$h[n] = 0 \quad \text{for} \quad n < 0$$

Right-sided sequence

  – Similarly, ***anti-causal*** can be defined as

$$h[n] = 0 \quad \text{for} \quad n > 0$$

Left-sided sequence

- An LTI system is defined to be ***stable***, if for every bounded input it produces a bounded output
  – Necessary condition: $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$
    - That is Fourier transform exists, and therefore z-transform include the unit circle in its region of converge

Z-Transform

- **Right-Sided Sequence**

  – E.g., the exponential signal

1. $h_1[n] = a^n u[n],\quad$ where $u[n] = \begin{cases} 1 & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$

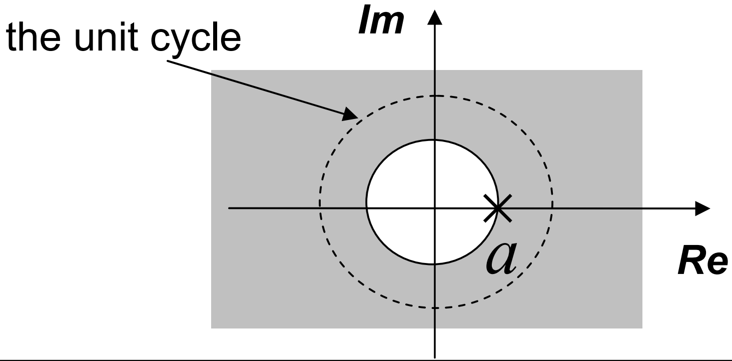
$$H_1(z) = \sum_{n=-\infty}^{\infty} a^n z^{-n} = \sum_{n=-\infty}^{\infty} \left(az^{-1}\right)^n = \frac{1}{1 - az^{-1}}$$

have a pole at $z = a$

(Pole: z-transform goes to infinity)

If $\left|az^{-1}\right| < 1$

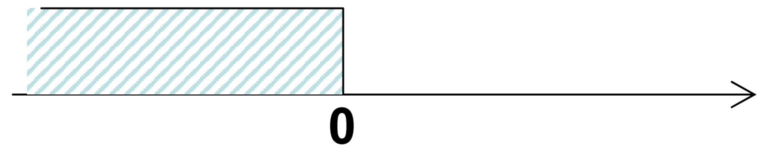
$$\therefore\ ROC_1 \text{ is } |z| > |a|$$



Fourier transform of $h_1[n]$ exists if $|a| < 1$

Z-Transform

- **Left-Sided Sequence**
  - E.g.



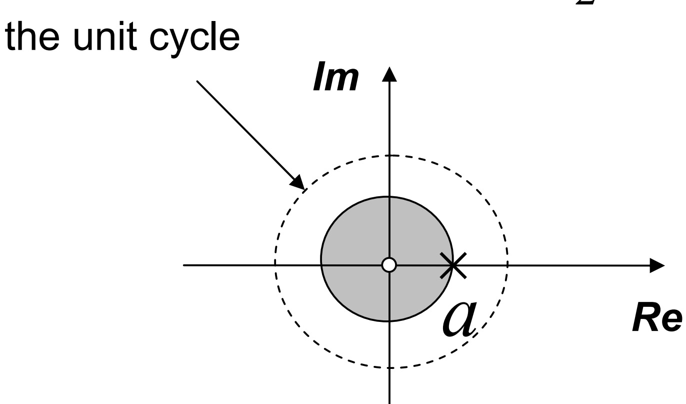
2. $h_2[n] = -a^n u[-n-1]$

$$H_2(z) = -\sum_{n=-\infty}^{\infty} a^n u[-n-1] z^{-n} = -\sum_{n=-\infty}^{-1} a^n z^{-n}$$

If $\left|a^{-1}z\right| < 1$

$$= -\sum_{n=1}^{\infty} a^{-n} z^{n} = 1 - \sum_{n=0}^{\infty} \left(a^{-1}z\right)^n = 1 - \frac{1}{1-a^{-1}z} = -\frac{-a^{-1}z}{1-a^{-1}z} = \frac{1}{1-az^{-1}}$$

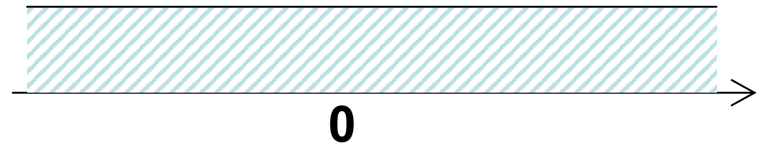
$$\therefore\ ROC_2 \text{ is } |z| < |a|$$



when $|a| < 1$, the Fourier transform of $h_2[n]$ doesn't exist, because $h_2[n]$ will go exponentially as $n \to -\infty$

Z-Transform

- **Two-Sided Sequence**



  – E.g.

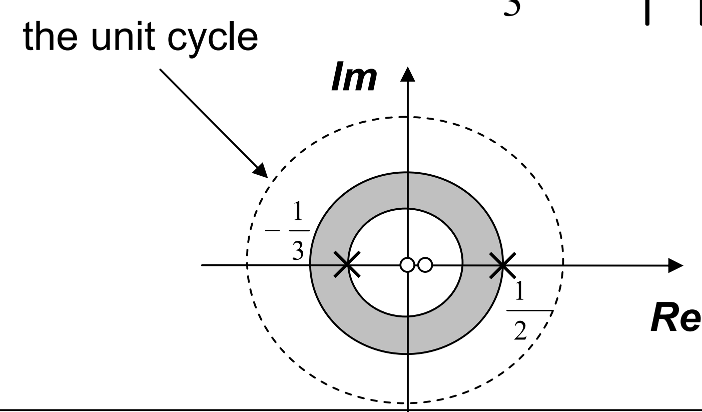
$$3.\quad h_3[n] = \left(-\frac{1}{3}\right)^n u[n] - \left(\frac{1}{2}\right)^n u[-n-1]$$

$$\left(-\frac{1}{3}\right)^n u[n] \quad \overset{z}{\longleftrightarrow} \quad \frac{1}{1+\frac{1}{3}z^{-1}}, \quad |z| > \frac{1}{3}$$

$$-\left(\frac{1}{2}\right)^n u[-n-1] \quad \overset{z}{\longleftrightarrow} \quad \frac{1}{1-\frac{1}{2}z^{-1}}, \quad |z| < \frac{1}{2}$$

$$\Rightarrow \quad H_3(z) = \frac{1}{1+\frac{1}{3}z^{-1}} + \frac{1}{1-\frac{1}{2}z^{-1}} = \frac{2z\left(z-\frac{1}{12}\right)}{\left(z+\frac{1}{3}\right)\left(z-\frac{1}{2}\right)}$$

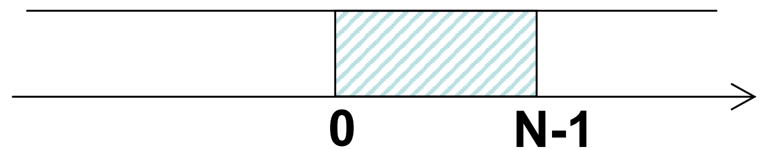
$$\therefore\ ROC_3 \text{ is } |z| < \left|\frac{1}{2}\right| \text{ and } |z| > \left|\frac{1}{3}\right|$$



Fourier transform of $h_3[n]$ doesn't exist, because ROC_3 doesn't include the unit circle

Z-Transform

- **Finite-length Sequence**



  – E.g.

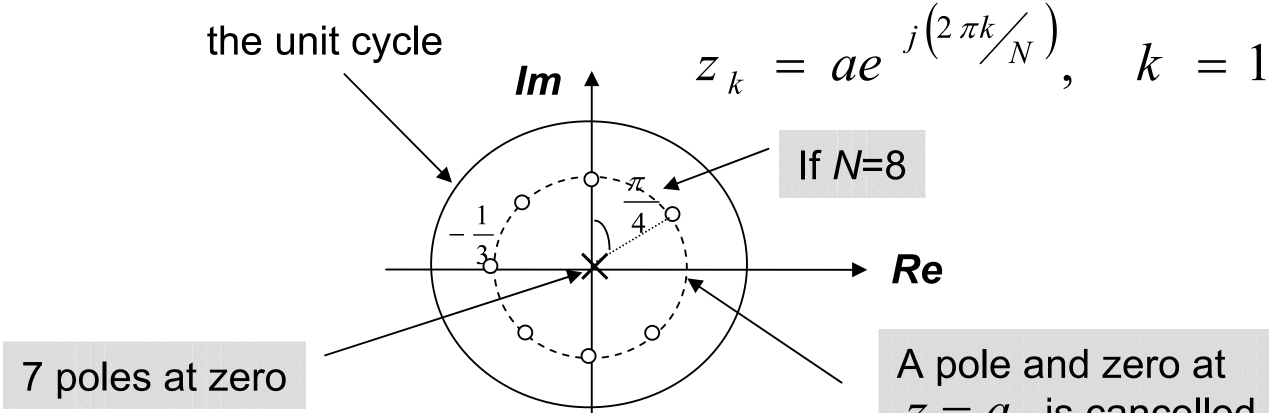
3. $h_4[n] = \begin{cases} a^n, & 0 \le n \le N-1 \\ 0, & \text{others} \end{cases}$

$$z^{N-1} + az^{N-2} + a^2 z^{N-3} + \ldots.. + a^{N-1}$$

$$H_4(z) = \sum_{n=0}^{N-1} a^n z^{-n} = \sum_{n=0}^{N-1} \left(az^{-1}\right)^n = \frac{1-\left(az^{-1}\right)^N}{1-az^{-1}} = \frac{1}{z^{N-1}} \frac{z^N - a^N}{z-a}$$

$$\therefore ROC_4 \text{ is entire } z\text{-plane except } z = 0$$

$$z_k = ae^{j\left(2\pi k/N\right)}, \quad k = 1, \ldots, N-1$$



Z-Transform

- **Properties of *z*-transform**

1. If $h[n]$ is right-sided sequence, i.e. $h[n]=0,\ n \le n_1$ and if *ROC* is the exterior of some circle, the **all finite** z for which $|z| > r_0$ will be in *ROC*
   - If $n_1 \ge 0$,*ROC* will include $z = \infty$

A causal sequence is right-sided with $n_1 \ge 0$
$\therefore$ ROC is the exterior of circle including $z = \infty$

2. If $h[n]$ is left-sided sequence, i.e. $h[n]=0,\ n \ge n_2$, the *ROC* is the interior of some circle,
   - If $n_2 < 0$,*ROC* will include $z = 0$
3. If $h[n]$ is two-sided sequence, the *ROC* is a **ring**
4. The ROC can't contain any poles

Summary of the Fourier and *z*-transforms

Table 5.5 Properties of the Fourier and *z*-transforms.

Property	Signal	Fourier Transform	*z*-Transform
Linearity	$ax_1[n]+bx_2[n]$	$aX_1(e^{j\omega})+bX_2(e^{j\omega})$	$aX_1(z)+bX_2(z)$
Symmetry	$x[-n]$	$X(e^{-j\omega})$	$X(z^{-1})$
	$x^*[n]$	$X^*(e^{-j\omega})$	$X^*(z^*)$
	$x^*[-n]$	$X^*(e^{j\omega})$	$X^*(1/z^*)$
	$x[n]$ real	$X(e^{j\omega})$ is *Hermitian* $X(e^{-j\omega})=X^*(e^{j\omega})$ $\lvert X(e^{j\omega})\rvert$ is even[6] $\mathrm{Re}\{X(e^{j\omega})\}$ is even $\arg\{X(e^{j\omega})\}$ is odd[7] $\mathrm{Im}\{X(e^{j\omega})\}$ is odd	$X(z^*)=X^*(z)$
	$\mathrm{Even}\{x[n]\}$	$\mathrm{Re}\{X(e^{j\omega})\}$	
	$\mathrm{Odd}\{x[n]\}$	$j\,\mathrm{Im}\{X(e^{j\omega})\}$	
Time-shifting	$x[n-n_0]$	$X(e^{j\omega})e^{-j\omega n_0}$	$X(z)z^{-n_0}$
Modulation	$x[n]e^{j\omega_0 n}$	$X(e^{j(\omega-\omega_0)})$	$X(e^{-j\omega_0}z)$
	$x[n]z_0^n$		$X(z/z_0)$
Convolution	$x[n]*h[n]$	$X(e^{j\omega})H(e^{j\omega})$	$X(z)H(z)$
	$x[n]y[n]$	$\frac{1}{2\pi}X(e^{j\omega})*Y(e^{j\omega})$	
Parseval's Theorem	$R_{xx}[n]=\sum_{m=-\infty}^{\infty}x[m+n]x^*[m]$	$S_{xx}(\omega)=\lvert X(\omega)\rvert^2$	$X(z)X^*(1/z^*)$

LTI Systems in the Frequency Domain

- **Example 1**: A complex exponential sequence $x[n] = e^{j\omega n}$
  - System impulse response $h[n]$

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} h[k] e^{j\omega(n-k)}$$

$$= e^{j\omega n} \sum_{k=-\infty}^{\infty} h[k] e^{-j\omega k}$$

$$= H(e^{j\omega}) e^{j\omega n}$$

$$\begin{aligned} y[n] &= x[n] * h[n] \\ &= h[n] * x[n] \\ &= \sum_{k=-\infty}^{\infty} x[k] h[n-k] \\ &= \sum_{k=-\infty}^{\infty} h[k] x[n-k] \end{aligned}$$

$H(e^{j\omega})$: the Fourier transform of the system impulse response. It is often referred to as the system frequency response.

  - Therefore, a complex exponential input to an LTI system results in the same complex exponential at the output, but modified by $H(e^{j\omega})$
    - The complex exponential is an eigenfunction of an LTI system, and $H(e^{j\omega})$ is the associated eigenvalue

$$T\{x[n]\} = H(e^{j\omega}) x[n]$$

LTI Systems in the Frequency Domain

- **Example 2**: A sinusoidal sequence $x[n] = A\cos(w_0 n + \phi)$

$$x[n] = A\cos(\omega_0 n + \phi)$$
$$= \frac{A}{2}e^{j\phi}e^{j\omega_0 n} + \frac{A}{2}e^{-j\phi}e^{-j\omega_0 n}$$

$$e^{j\theta} = \cos\theta + i\sin\theta$$
$$e^{-j\theta} = \cos\theta - i\sin\theta$$
$$\Rightarrow \cos\theta = \frac{1}{2}\left(e^{j\theta} + e^{-j\theta}\right)$$

  - System impulse response $h[n]$

$$z = x + jy \qquad \Rightarrow e^{j\omega} = \cos\omega + j\sin\omega$$
$$z^* = x - jy \qquad \Rightarrow \left(e^{j\omega}\right)^* = \cos\omega - j\sin\omega$$
$$e^{-j\omega} = \cos(-\omega) + j\sin(-\omega)$$
$$= \cos\omega - j\sin\omega$$

$$y[n] = H\left(e^{j\omega_0}\right)\frac{A}{2}e^{j\phi}e^{j\omega_0 n} + H\left(e^{-j\omega_0}\right)\frac{A}{2}e^{-j\phi}e^{-j\omega_0 n}$$

$$H\left(e^{-j\omega_0}\right) = H^*\left(e^{j\omega_0}\right)$$
$$H^*\left(e^{j\omega_0}\right) = \left|H\left(e^{j\omega_0}\right)\right|e^{-j\angle H\left(e^{j\omega_0}\right)}$$

$$= \frac{A}{2}\left[H\left(e^{j\omega_0}\right)e^{j(\omega_0 n+\phi)} + H^*\left(e^{j\omega_0}\right)e^{-j(\omega_0 n+\phi)}\right]$$

$$= \frac{A}{2}\left[\left|H\left(e^{j\omega_0}\right)\right|e^{j\angle H\left(e^{j\omega_0}\right)}e^{j(\omega_0 n+\phi)} + \left|H\left(e^{j\omega_0}\right)\right|e^{-j\angle H\left(e^{j\omega_0}\right)}e^{-j(\omega_0 n+\phi)}\right]$$

$$= A\left|H\left(e^{j\omega_0}\right)\right|\cos\left[\omega_0 n + \phi + \angle H\left(e^{j\omega_0}\right)\right]$$

LTI Systems in the Frequency Domain

- **Example 3**: A sum of sinusoidal sequences

$$x[n] = \sum_{k=1}^{K} A_k \cos(\omega_k n + \phi_k)$$

$$y[n] = \sum_{k=1}^{K} A_k \left| H\left(e^{j\omega_k}\right) \right| \cos\left[\omega_k n + \phi_k + \angle H\left(e^{j\omega_k}\right)\right]$$

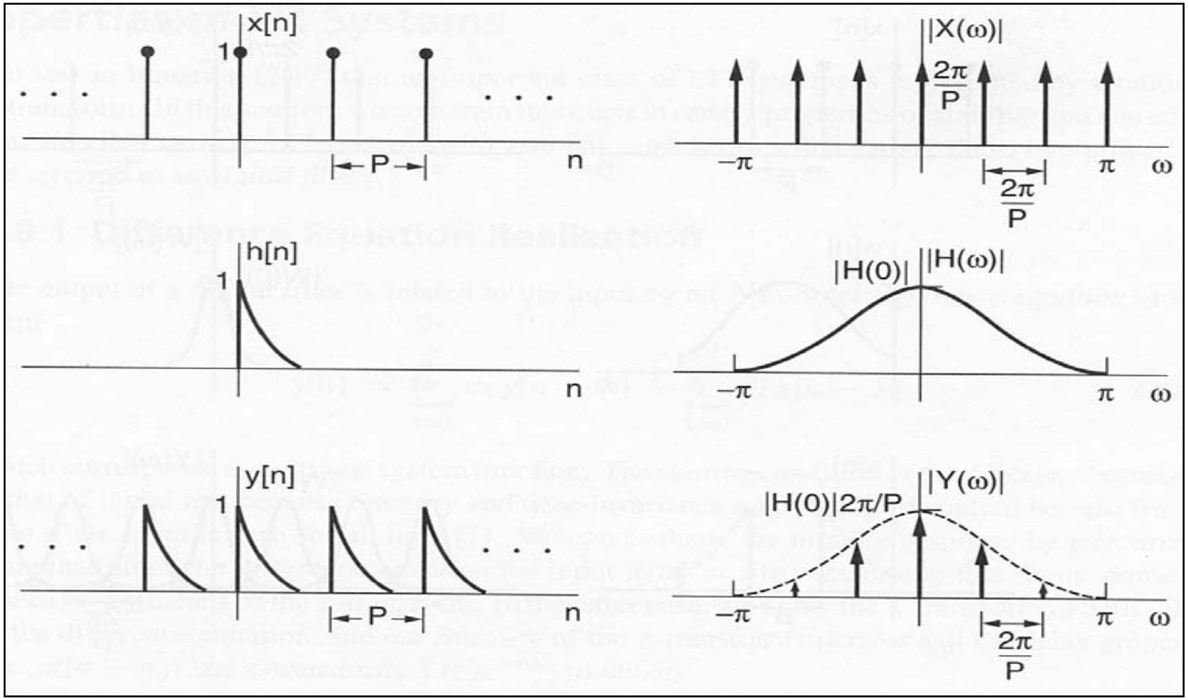
  – A similar expression is obtained for an input consisting of a sum of complex exponentials

LTI Systems in the Frequency Domain

- **Example 4: Convolution Theorem** $x[n]*h[n] \Leftrightarrow X(e^{j\omega})H(e^{j\omega})$

$$x[n] = \sum_{k=-\infty}^{\infty} \delta[n-kP] \quad \xrightarrow{\textbf{DTFT}} \quad X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} \frac{2\pi}{P} \delta\left[\left(\omega - \frac{2\pi}{P}k\right)\right]$$

$$h[n] = \sum_{k=-\infty}^{\infty} a^n u[n], \ |a| < 1 \quad \xrightarrow{\textbf{DTFT}} \quad H(e^{j\omega}) = \frac{1}{1-ae^{-j\omega}}$$



$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$$

$$= \frac{1}{1-ae^{-j\omega}} \sum_{k=-\infty}^{\infty} \frac{2\pi}{P} \delta\left[\left(\omega - \frac{2\pi}{P}k\right)\right]$$

$$= \frac{2\pi}{P} \sum_{k=-\infty}^{\infty} \frac{1}{1-ae^{-j\frac{2\pi}{P}k}} \delta\left[\left(\omega - \frac{2\pi}{P}k\right)\right]$$

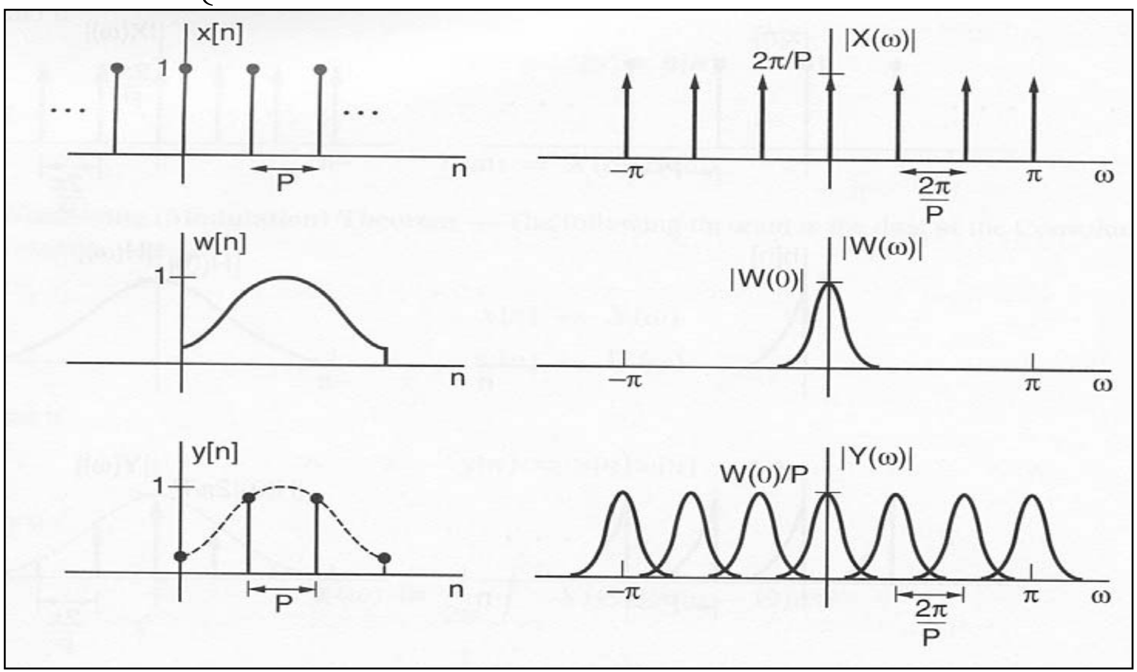
LTI Systems in the Frequency Domain

- **Example 5: Windowing Theorem** $x[n]w[n] \Leftrightarrow \frac{1}{2\pi} W\left(e^{j\omega}\right) * X\left(e^{j\omega}\right)$

$$x[n] = \sum_{k=-\infty}^{\infty} \delta[n - kP]$$

$$w[n] = \begin{cases} 0.54 - 0.46\cos\left(\frac{2\pi n}{N-1}\right), & n = 0,1,\ldots\ldots,N-1 \\ 0 & \text{otherwise} \end{cases}$$

Hamming window



$$Y\left(e^{j\omega}\right) = \frac{1}{2\pi} W\left(e^{j\omega}\right) * X\left(e^{j\omega}\right)$$

$$= \frac{1}{2\pi} W\left(e^{j\omega}\right) * \sum_{k=-\infty}^{\infty} \frac{2\pi}{P} \delta\left(\omega - \frac{2\pi}{P}k\right)$$

$$= \frac{1}{P} \sum_{k=-\infty}^{\infty} \left\{ W\left(e^{j\omega}\right) * \delta\left(\omega - \frac{2\pi}{P}k\right) \right\}$$

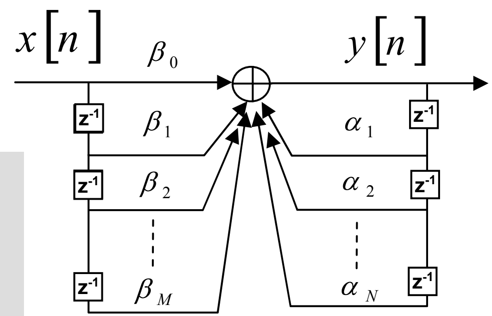
$$= \frac{1}{P} \sum_{k=-\infty}^{\infty} \left\{ \sum_{m=-\infty}^{m=\infty} W\left(e^{jm}\right) \delta\left(\omega - \frac{2\pi}{P}k - m\right) \right\}$$

$$= \frac{1}{P} \sum_{k=-\infty}^{\infty} W\left(e^{j\left(\omega - \frac{2\pi}{P}k\right)}\right)$$

Difference Equation Realization for a Digital Filter

- The relation between the output and input of a digital filter can be expressed by

$$y[n] = \sum_{k=1}^{N} \alpha_k y[n-k] + \sum_{k=0}^{M} \beta_k x[n-k]$$



linearity and delay properties

delay property

$$x[n] \rightarrow X(z)$$
$$x[n-n_0] \rightarrow X(z) z^{-n_0}$$

$$Y(z) = \sum_{k=1}^{N} \alpha_k Y(z) z^{-k} + \sum_{k=0}^{M} \beta_k X(z) z^{-k}$$

A rational transfer function

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} \beta_k z^{-k}}{1 - \sum_{k=1}^{N} \alpha_k z^{-k}}$$

Causal:
Rightsided, the ROC outside the outmost pole
Stable:
The ROC includes the unit circle
Causal and Stable:
all poles must fall inside the unit circle (not including zeros)

Difference Equation Realization for a Digital Filter

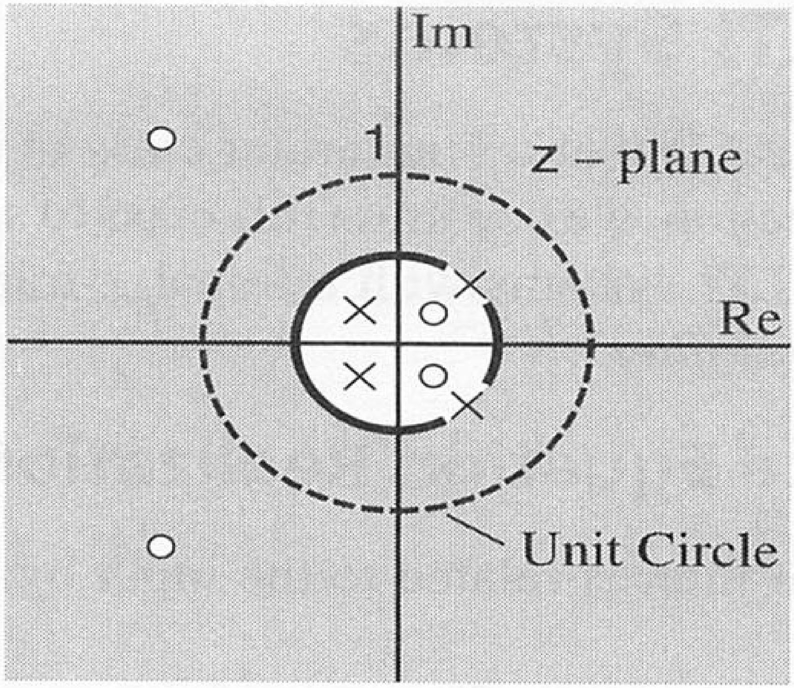


Figure 2.8 Pole-zero configuration for a causal and stable discrete-time system.

Magnitude-Phase Relationship

- **Minimum phase system:**
  - The z-transform of a system impulse response sequence (a rational transfer function) has all zeros as well as poles inside the unit cycle
  - Poles and zeros called “minimum phase components”
  - **Maximum phase**: all zeros (or poles) outside the unit cycle
- **All-pass system:**

$$\left[\frac{1\text{-}a^{*}z}{1-az^{-1}}\right]^{\pm 1}$$

  - Consist a cascade of factor of the form

Poles and zeros occur at conjugate reciprocal locations

  - Characterized by a frequency response with unit (or flat) magnitude for all frequencies

$$\left|\frac{1\text{-}a^{*}z}{1-az^{-1}}\right| = 1$$

Magnitude-Phase Relationship

- Any digital filter can be represented by the cascade of a minimum-phase system and an all-pass system

$$H(z) = H_{\min}(z) H_{ap}(z)$$

Suppose that $H(z)$ has only one zero $1/a^*$ $(|a| < 1)$ outside the unit circle. $H(z)$ can be expressed as:

$$H(z) = H_1(z)(1 - a^* z) \quad (H_1(z) \text{ is a minimum phase filter})$$

$$= H_1(z)(1 - az^{-1}) \frac{(1 - a^* z)}{(1 - az^{-1})}$$

where:

$H_1(z)(1 - az^{-1})$ is also a minimum phase filter.

$\frac{(1 - a^* z)}{(1 - az^{-1})}$ is a all-pass filter.

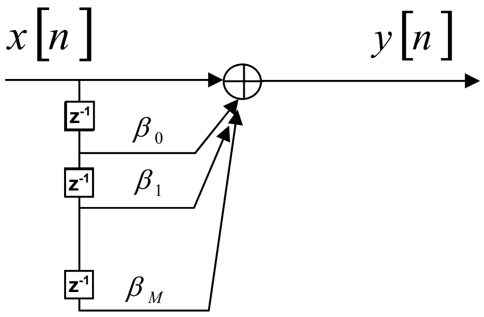
FIR Filters

- FIR (Finite Impulse Response)
  - The impulse response of an FIR filter has finite duration
  - Have no denominator in the rational function $H(z)$
    - No feedback in the difference equation

$$y[n] = \sum_{r=0}^{M} \beta_r x[n-r]$$

$$h[n] = \begin{cases} \beta_n, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$$

$$H(z) = \frac{Y(z)}{X(z)} = \sum_{k=0}^{M} \beta_k z^{-k}$$



  - Can be implemented with simple a train of delay, multiple, and add operations

First-Order FIR Filters

- A special case of FIR filters

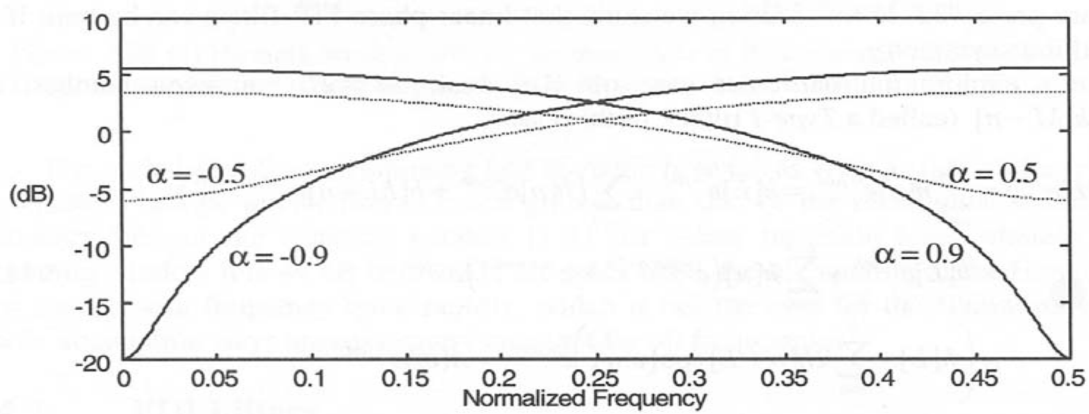
$$y[n] = x[n] + \alpha x[n-1] \quad \Longleftrightarrow \quad H(z) = 1 + \alpha z^{-1}$$

$$H(e^{j\omega}) = 1 + \alpha e^{-j\omega}$$

$$\left|H(e^{j\omega})\right|^2 = \left|1 + \alpha(\cos\omega - j\sin\omega)\right|^2$$

$$= (1 + \alpha\cos\omega)^2 + (\alpha\sin\omega)^2 = 1 + 2\alpha\cos\omega$$

$$\theta(e^{j\omega}) = -\arctan\left(\frac{\alpha\sin\omega}{1 + \alpha\cos\omega}\right)$$



$$10\log\left|H(e^{j\omega})\right|^2$$

Figure 5.21 Frequency response of the first order FIR filter for various values of α.

Discrete Fourier Transform (DFT)

- The Fourier transform of a discrete-time sequence is a continuous function of frequency
  - We need to sample the Fourier transform finely enough to be able to recover the sequence
  - For a sequence of finite length *N*, sampling yields the new transform referred to as *discrete Fourier transform* (DFT)

$$X(k) = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn}, \quad 0 \le n \le N-1$$

DFT, **Analysis**

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi}{N}kn}, \quad 0 \le n \le N-1$$

Inverse DFT, **Synthesis**

Discrete Fourier Transform (DFT)

$$\forall\ 0 \le k \le N-1$$

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn}, \quad 0 \le n \le N-1$$

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & e^{-j\frac{2\pi}{N}1\cdot 1} & \cdots & e^{-j\frac{2\pi}{N}1\cdot(N-1)} \\ \vdots & \vdots & \cdots & \vdots \\ 1 & e^{-j\frac{2\pi}{N}(N-1)\cdot 1} & \cdots & e^{-j\frac{2\pi}{N}(N-1)\cdot(N-1)} \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix} = \begin{bmatrix} X[0] \\ X[1] \\ \vdots \\ X[N-1] \end{bmatrix}$$

Discrete Fourier Transform (DFT)

- Orthogonality of Complex Exponentials

$$\frac{1}{N}\sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}(k-r)n} = \begin{cases} 1, & \text{if } k\text{-}r = mN \\ 0, & \text{otherwise} \end{cases}$$

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi}{N}kn}$$

$$\Rightarrow \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}rn} = \sum_{n=0}^{N-1} \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi}{N}(k-r)n}$$

$$= \sum_{k=0}^{N-1} X[k] \left[\frac{1}{N} \sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}(k-r)n} \right]$$

$$X[k] = X[r + mN] = X[r]$$

$$= X[r]$$

$$\Rightarrow X[k] = \sum_{r=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn}$$

Discrete Fourier Transform (DFT)

- Parseval's theorem

$$\sum_{n=0}^{N-1} \left| x[n] \right|^2 = \frac{1}{N} \sum_{k=0}^{N-1} \left| X(k) \right|^2$$

Energy density