Further Topics on Random Variables: Derived Distributions

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Reference:

- D. P. Bertsekas, J. N. Tsitsiklis, *Introduction to Probability* , Section 4.1

Two-step approach to Calculating Derived PDF

- Calculate the PDF of a Function $Y = g(X)$ of a continuous random variable X

1. Calculate the CDF F_Y of Y using the formula

$$F_Y(y) = \mathbf{P}(g(X) \leq y) = \int_{\{x | g(x) \leq y\}} f_X(x) dx$$

2. Differentiate to obtain the PDF (called the derived distribution) of Y

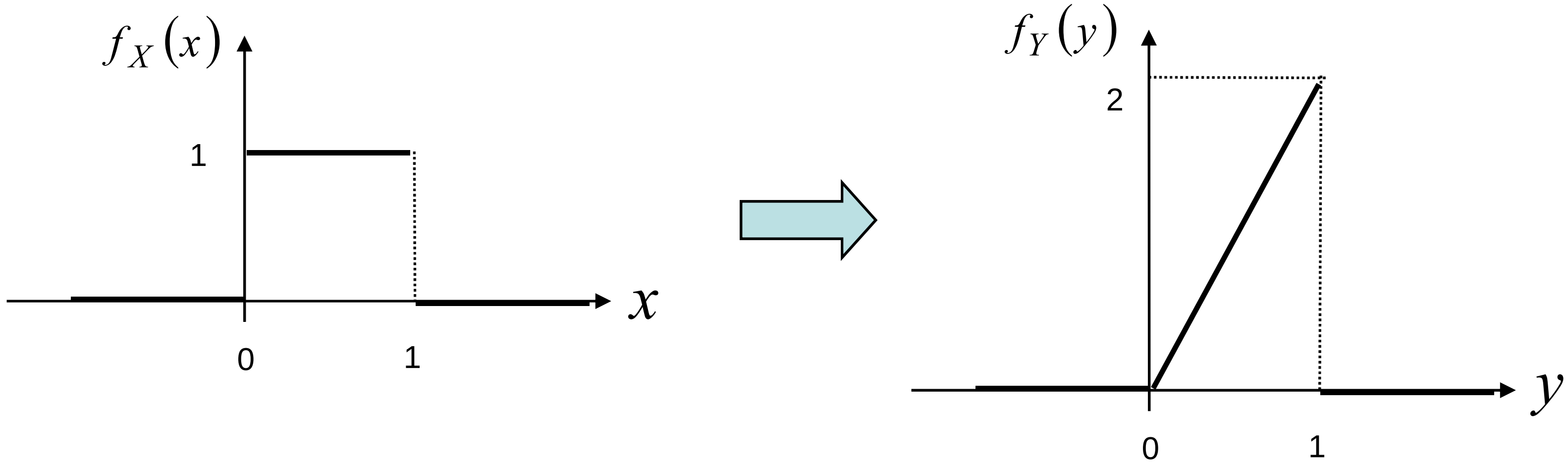
$$f_Y(y) = \frac{dF_Y(y)}{dy}$$

Illustrative Examples (1/2)

- **Example 4.1.** Let X be uniform on [0, 1]. Find the PDF of $Y = \sqrt{X}$. Note that Y takes values between 0 and 1.

$$F_Y(y) = \mathbf{P}(Y \le y) = \mathbf{P}\left(\sqrt{X} \le y\right) = \mathbf{P}\left(X \le y^2\right) = y^2$$

$$\therefore\ f_Y(y) = \frac{dF_Y(y)}{dy} = 2y, \qquad 0 \le y \le 1$$



Illustrative Examples (2/2)

- **Example 4.3.** Let $Y = X^2$, where X is a random variable with known PDF $f_X(x)$. Find the PDF of Y represented in terms of $f_X(x)$.

For any $y \geq 0$, we have

$$
\begin{aligned}
F_Y(y) &= \mathbf{P}(Y \leq y) = \mathbf{P}\left(X^2 \leq y\right) \\
&= \mathbf{P}\left(-\sqrt{y} \leq X \leq \sqrt{y}\right) \\
&= F_X\left(\sqrt{y}\right) - F_X\left(-\sqrt{y}\right)
\end{aligned}
$$

⇨

$$
\begin{aligned}
f_Y(y) &= \frac{dF_Y(y)}{dy} \\
&= \left[\frac{dF_X\left(\sqrt{y}\right)}{d\sqrt{y}} \cdot \frac{d\sqrt{y}}{dy}\right] - \left[\frac{dF_X\left(-\sqrt{y}\right)}{d\left(-\sqrt{y}\right)} \cdot \frac{d\left(-\sqrt{y}\right)}{dy}\right] \\
&= \frac{1}{2\sqrt{y}} f_X\left(\sqrt{y}\right) + \frac{1}{2\sqrt{y}} f_X\left(-\sqrt{y}\right) \\
&= \frac{1}{2\sqrt{y}} \left[f_X\left(\sqrt{y}\right) + f_X\left(-\sqrt{y}\right)\right]
\end{aligned}
$$

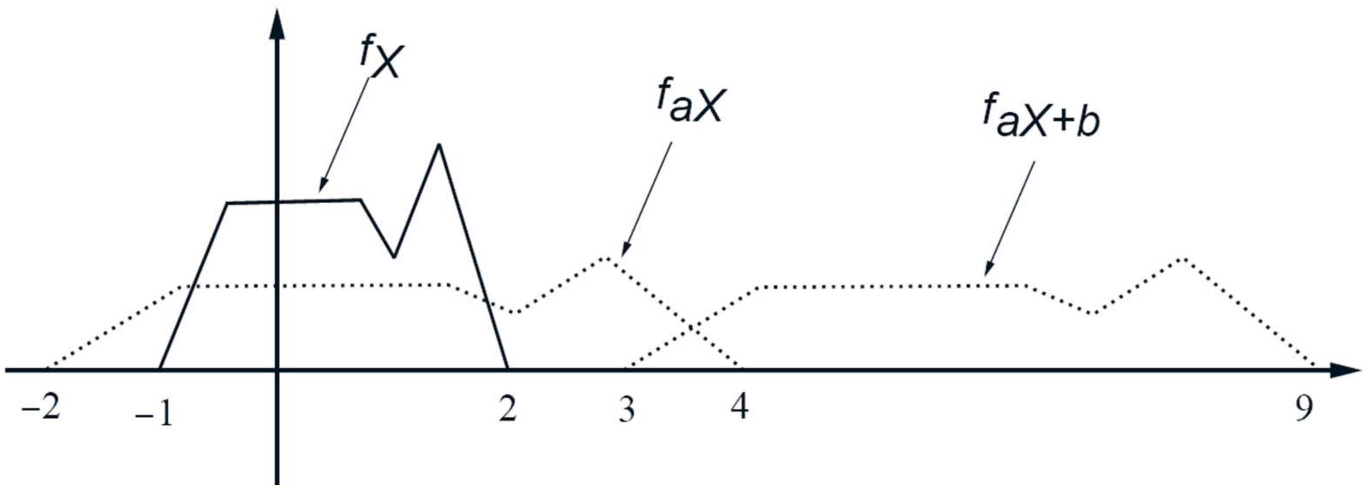
The PDF of a Linear Function of a Random Variable

- Let X be a continuous random variable with PDF $f_X(x)$, and let

$$Y = aX + b,$$

for some scalar $a \neq 0$ and b. Then,

$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$$



a>0, b>0

The PDF of a Linear Function of a Random Variable (1/2)

- Verification of the above formula

$$F_Y(y) = \mathbf{P}(Y \le y) = \mathbf{P}(aX + b \le y)$$

(i) For $a > 0$

$$F_Y(y) = \mathbf{P}\left(X \le \frac{y-b}{a}\right) = F_X\left(\frac{y-b}{a}\right)$$

$$\Rightarrow f_Y(y) = \frac{dF_X\left(\frac{y-b}{a}\right)}{d\left(\frac{y-b}{a}\right)} \cdot \frac{d\left(\frac{y-b}{a}\right)}{dy} = \frac{1}{a} f_X\left(\frac{y-b}{a}\right)$$

(ii) For $a < 0$

$$F_Y(y) = \mathbf{P}\left(X \ge \frac{y-b}{a}\right) = 1 - F_X\left(\frac{y-b}{a}\right)$$

$$\Rightarrow f_Y(y) = -\frac{dF_X\left(\frac{y-b}{a}\right)}{d\left(\frac{y-b}{a}\right)} \cdot \frac{d\left(\frac{y-b}{a}\right)}{dy} = -\frac{1}{a} f_X\left(\frac{y-b}{a}\right)$$

$$\therefore f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$$

Illustrative Examples (1/2)

- **Example 4.4. A linear function of an exponential random variable.**
  - Suppose that X is an exponential random variable with PDF

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

    - where λ is a positive parameter. Let $Y = aX + b$. Then,

$$f_Y(y) = \begin{cases} \frac{1}{|a|} \lambda e^{-\lambda (y-b)/a}, & \text{if } (y-b)/a \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Note that if $a > 0$ and $b = 0$, then Y is a exponential with parameter λ / a

Illustrative Examples (2/2)

- **Example 4.5. A linear function of a normal random variable is normal.**
  - Suppose that X is a normal random variable with mean μ and variance σ^2,

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad -\infty \le x \le \infty$$

  - And let $Y = aX + b$, where a and b are some scalars. We have

$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$$

$$= \frac{1}{|a|} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{\left(\frac{y-b}{a}-\mu\right)^2}{2\sigma^2}}$$

$\therefore Y$ is also a normal random variable with mean $a\mu + b$ and variance $a^2\sigma^2$

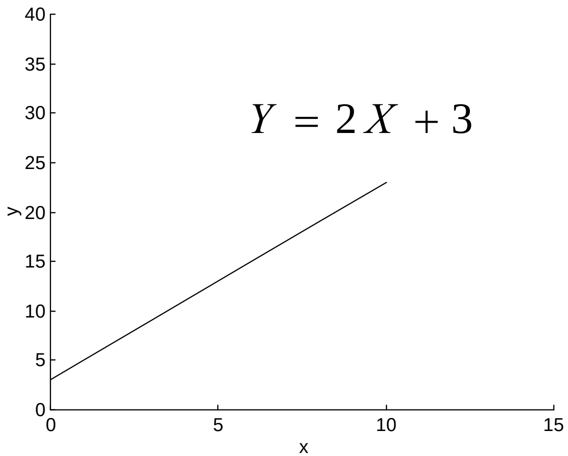
$$== \frac{1}{\sqrt{2\pi}|a|\sigma} e^{-\frac{(y-(b+a\mu))^2}{2a^2\sigma^2}}, \; -\infty \le y \le \infty$$

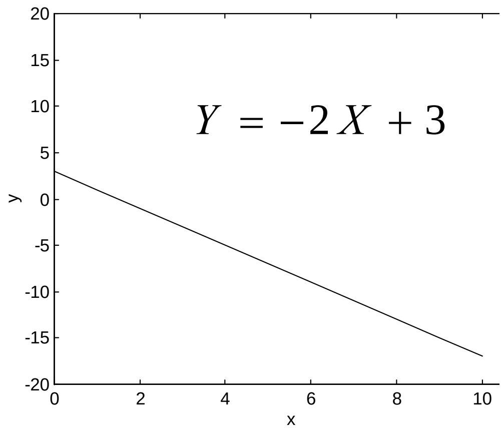
Monotonic Functions of a Random Variable (1/4)

- Let X be a continuous random variable and have values in a certain interval I ($f_X(x)=0$ for $x \notin I$). While random variable $Y = g(X)$ and we assume that g is **strictly monotonic** over the interval I . That is, either

  (1) $g(x) < g(x')$ for all $x, x' \in I$, satisfying $x < x'$
  (monotonically increasing case), or

  (2) $g(x) > g(x')$ for all $x, x' \in I$, satisfying $x < x'$
  (monotonically decreasing case)





Monotonic Functions of a Random Variable (2/4)

- Suppose that g is monotonic and that for some function h and all x in the range I of X we have

$$y = g(x) \quad \text{if and only if} \quad x = h(y)$$

  - For example,

$$y = g(x) = ax + b \qquad \Rightarrow \qquad x = h(y) = \frac{y-b}{a}$$

$$y = g(x) = e^{ax} \qquad \Rightarrow \qquad x = h(y) = \frac{\ln y}{a}$$

$$y = g(x) = -ax + b \qquad \Rightarrow \qquad x = h(y) = -\frac{y-b}{a}$$

Monotonic Functions of a Random Variable (3/4)

– Assume that h has first derivative $\frac{dh(y)}{dy}$. Then the PDF of Y in the region where $f_Y(y) > 0$ is given by

$$f_Y(y) = f_X(h(y))\left|\frac{dh(y)}{dy}\right|$$

- For the monotonically increasing case

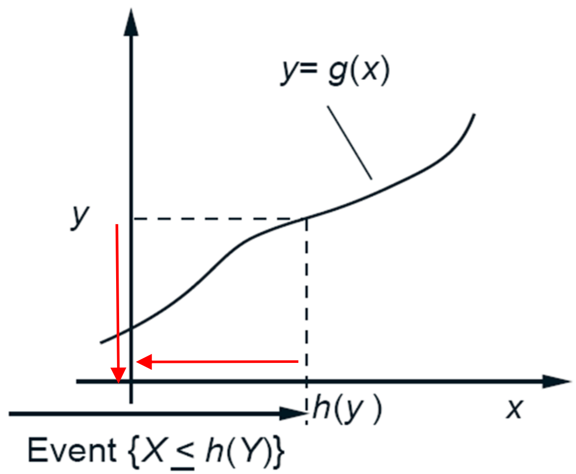
$$F_Y(y) = \mathbf{P}(g(X) \le y) = \mathbf{P}(X \le h(y)) = F_X(h(y))$$

$$\Rightarrow f_Y(y) = \frac{dF_Y(y)}{dy}$$

$$= \frac{dF_X(h(y))}{dy} = \frac{dF_X(h(y))}{dh(y)} \cdot \frac{dh(y)}{dy}$$

$$= f_X(h(y)) \cdot \frac{dh(y)}{dy}$$

$$\frac{dh(y)}{dy} > 0$$



Monotonic Functions of a Random Variable (4/4)

- For the monotonically decreasing case

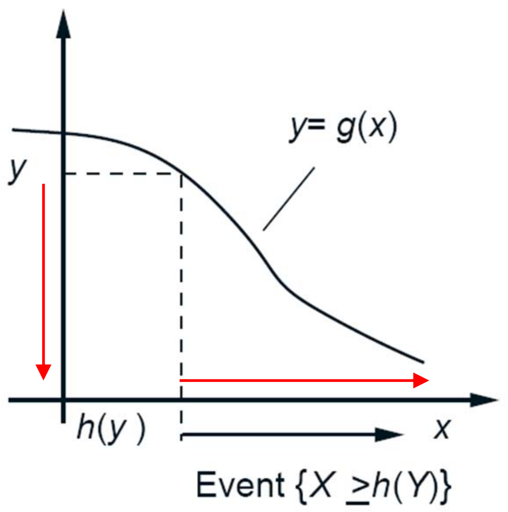
$$F_Y(y) = \mathbf{P}(g(X) \le y) = \mathbf{P}(X \ge h(y)) = 1 - F_X(h(y))$$

$$\Rightarrow f_Y(y) = \frac{dF_Y(y)}{dy}$$

$$= -\frac{dF_X(h(y))}{dy} = -\frac{dF_X(h(y))}{dh(y)} \cdot \frac{dh(y)}{dy}$$

$$= -f_X(h(y)) \cdot \frac{dh(y)}{dy}$$

$$\frac{dh(y)}{dy} < 0$$



Illustrative Examples (1/5)

- **Example 4.6.** Let $Y = g(X) = X^2$, where X is a continuous uniform random variable in the interval $(0, 1]$.
  - What is the PDF of y ?
    - Within this interval, g is strictly monotonic, and its inverse is $h(y) = \sqrt{y}$

We have

$$f_X(x) = 1 \quad \text{for all} \quad 0 < x \le 1$$

and $g(X)$ being strictly increasing

$\Rightarrow$

$$f_X(\sqrt{y}) = 1, \qquad \text{for all } 0 < y \le 1$$

$$\therefore \quad f_Y(y) = \frac{dh(y)}{dy} f_X(\sqrt{y}) = \begin{cases} \dfrac{1}{2\sqrt{y}}, & \text{if } y \in (0, 1] \\ 0, & \text{otherwise} \end{cases}$$

Illustrative Examples (2/5)

- **Example 4.7.** Let X and Y be independent random variables that are uniformly distributed on the interval [0, 1], respectively. What is the PDF of the random variable $Z = \max\{X, Y\}$

$$\begin{aligned} F_Z(z) &= \mathbf{P}\left(\max\{X, Y\} \leq z\right) \\ &= \mathbf{P}\left(X \leq z, Y \leq z\right) \\ &= \mathbf{P}\left(X \leq z\right)\mathbf{P}\left(Y \leq z\right) \\ &= z^2 \end{aligned}$$

$$\therefore\ f_Z(z) = \begin{cases} 2z, & \text{if } 0 \leq z \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

Question

- Let X and Y be independent random variables that are uniformly distributed on the interval [0, 1], respectively. What is the PDF of the random variable $Z = \min\{X, Y\}$

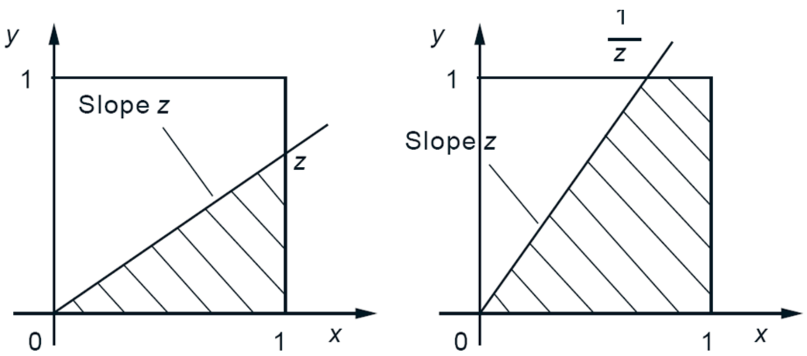
Illustrative Examples (3/5)

- **Example 4.8.** Let X and Y be independent random variables that are uniformly distributed on the interval [0, 1]. What is the PDF of the random variable $Z = Y / X$

$\because X, Y$ are independent

$\therefore f_{X,Y}(x, y) = f_X(x) f_Y(y) = 1,$

for all $x, y,\ 0 \le x, y \le 1$



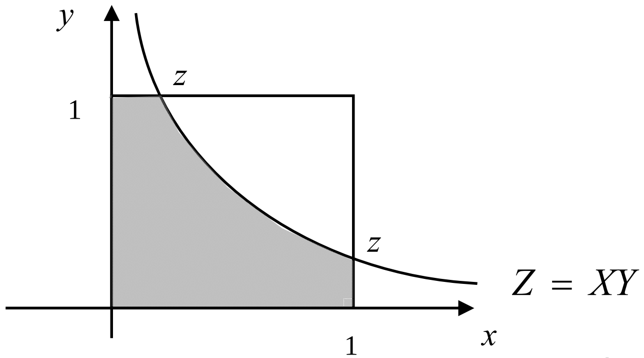
$$F_Z(z) = \mathbf{P}(Y / X \le z)$$

$$= \begin{cases} z/2, & \text{if } 0 \le z \le 1 \\ 1 - (1/2z), & \text{if } z > 1 \\ 0, & \text{otherwise} \end{cases}$$

$$\Rightarrow \quad f_Z(z) = \begin{cases} 1/2, & \text{if } 0 \le z \le 1 \\ 1/(2z^2), & \text{if } z > 1 \\ 0, & \text{otherwise} \end{cases}$$

Illustrative Examples (4/5)

- **Extra Example.** Let X and Y be independent random variables that are uniformly distributed on the interval [0, 1], respectively. What is the PDF of the random variable $Z = XY$



for $0 < z \le 1$

$$F_Z(z) = P(XY \le z)$$

$$= \int_0^z \int_0^1 f_{X,Y}(x,y)\,dy\,dx + \int_z^1 \int_0^{\frac{z}{x}} f_{X,Y}(x,y)\,dy\,dx$$

$$= \int_0^z \int_0^1 1\,dy\,dz + \int_z^1 \frac{z}{x}\,dx$$

$$= z + z \ln x \Big|_z^1$$

$$= z - z \ln z$$

$\Rightarrow$ for $0 < z \le 1$,

$$f_Z(z) = -\ln z$$

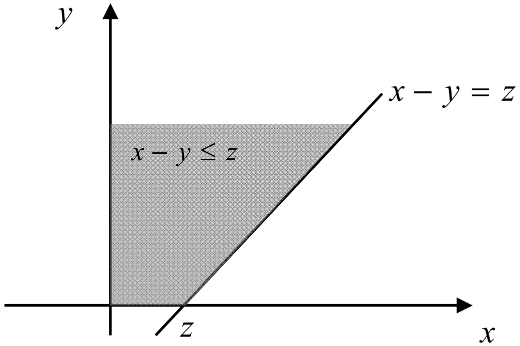
For example,

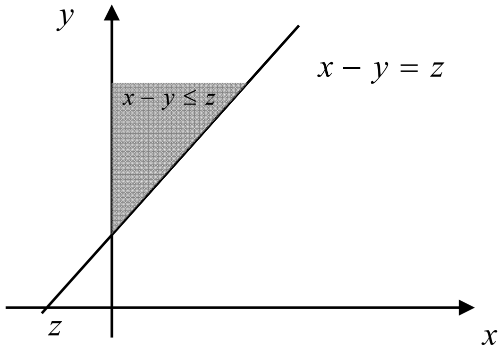
$$F_Z(1/8) = P(XY \le 1/8)$$

$$= \frac{1}{8} - \frac{1}{8}\ln\frac{1}{8} = \frac{1}{8} + \frac{3}{8}\ln 2$$

Illustrative Examples (5/5)

- **Example 4.9.** Let X and Y be independent random variables that are exponential distributed with parameter λ. What is the PDF of the random variable $Z = X - Y$





for $z \geq 0$

$$
\begin{aligned}
F_Z(z) &= P(X - Y \leq z) \\
&= \int_0^\infty \int_0^{y+z} f_{X,Y}(x,y)\,dxdy \\
&= \int_0^\infty \int_0^{y+z} \lambda e^{-\lambda y} \lambda e^{-\lambda x}\,dxdy \\
&= \int_0^\infty \lambda e^{-\lambda y} \left(\int_0^{y+z} \lambda e^{-\lambda x} dx \right) dy \\
&= \int_0^\infty \lambda e^{-\lambda y} \left(- e^{-\lambda x} \Big|_0^{y+z} \right) dy \\
&= \int_0^\infty \lambda e^{-\lambda y} dy - \int_0^\infty e^{-\lambda z} \lambda e^{-2\lambda y} dy \\
&= 1 - \frac{1}{2} e^{-\lambda z}
\end{aligned}
$$

$\Rightarrow$ for $z \geq 0$,

$$f_Z(z) = \frac{1}{2} e^{-\lambda z}$$

for $z \leq 0$

$$
\begin{aligned}
F_Z(z) &= P(X - Y \leq z) \\
&= \int_{-z}^\infty \int_0^{y+z} f_{X,Y}(x,y)\,dxdy \\
&= \int_{-z}^\infty \int_0^{y+z} \lambda e^{-\lambda y} \lambda e^{-\lambda x}\,dxdy \\
&= \int_{-z}^\infty \lambda e^{-\lambda y} \left(\int_0^{y+z} \lambda e^{-\lambda x} dx \right) dy \\
&= \int_{-z}^\infty \lambda e^{-\lambda y} \left(- e^{-\lambda x} \Big|_0^{y+z} \right) dy \\
&= \int_{-z}^\infty \lambda e^{-\lambda y} dy - \frac{1}{2} e^{-\lambda z} \int_{-z}^\infty 2\lambda e^{-2\lambda y} dy \\
&= e^{\lambda z} - \frac{1}{2} e^{-\lambda z} e^{2\lambda z} \\
&= \frac{1}{2} e^{\lambda z}
\end{aligned}
$$

$\Rightarrow$ for $z \leq 0$,

$$f_Z(z) = \frac{1}{2} e^{\lambda z}$$

An Extra Example

- Let X and Y be independent random variables that are uniformly distributed on the interval [0, 1], respectively. What is the PDF of the random variable $Z = \max\{2X, Y\}$

Let $X' = 2X \Rightarrow X'$ is uniformly distributed on $[0, 2]$

with PDF $f_{X'}(x') = 1/2$

$$F_Z(z) = \mathbf{P}(\max\{2X, Y\} \le z) = \mathbf{P}(\max\{X', Y\} \le z)$$
$$= \mathbf{P}(X' \le z, Y \le z)$$
$$= \mathbf{P}(X' \le z)\mathbf{P}(Y \le z)$$

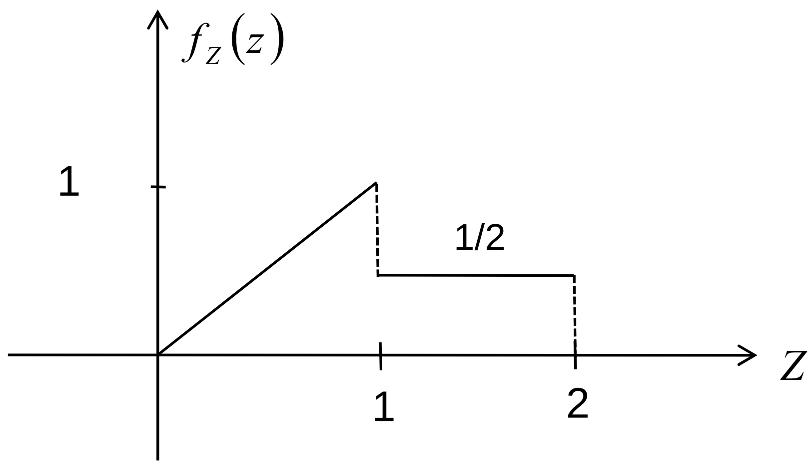
if $0 \le z \le 1$

$$\Rightarrow F_Z(z) = 1/2\, z \cdot z = 1/2\, z^2$$

if $1 \le z \le 2$

$$\Rightarrow F_Z(z) = 1/2\, z \cdot 1 = 1/2\, z$$

$$\therefore f_Z(z) = \begin{cases} z, & \text{if } 0 \le z \le 1 \\ 1/2, & \text{if } 1 \le z \le 2 \\ 0, & \text{otherwise} \end{cases}$$



Exercise

1. Let X and Y be independent random variables that are uniformly distributed on the interval [0, 1]. What is the PDF of the random variable $Z = 2X/3Y$

2. Let X and Y be independent random variables that are uniformly distributed on the interval [0, 1]. What is the PDF of the random variable $Z = 2X - 3Y$

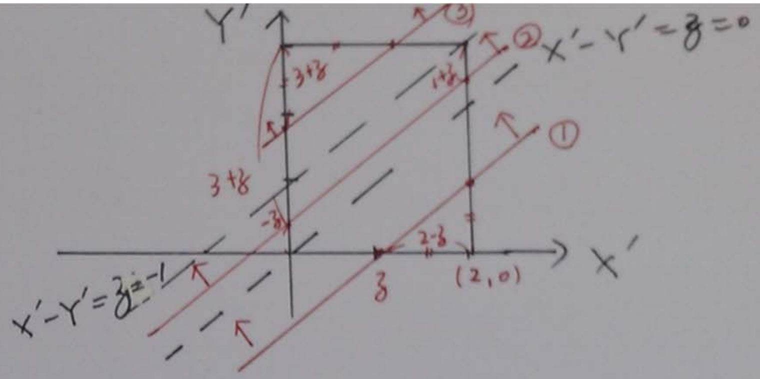
$X, Y \sim \text{Uniform } [0, 1]$

$Z = 2X - 3Y = X' - Y'$

$X' \sim \text{Uniform } [0, 2]$

$Y' \sim \text{Uniform } [0, 3]$

$f_{X'Y'}(x, y) = \frac{1}{6} \quad 0 \le x \le 2, \; 0 \le y \le 3$



① $0 \le z \le 2$

$F_Z(z) = \frac{1}{6}\left[6 - \frac{1}{2}(2-z)^2\right]$

$\Rightarrow f_Z(z) = \frac{1}{6}(2-z)$

② $-1 \le z \le 0$

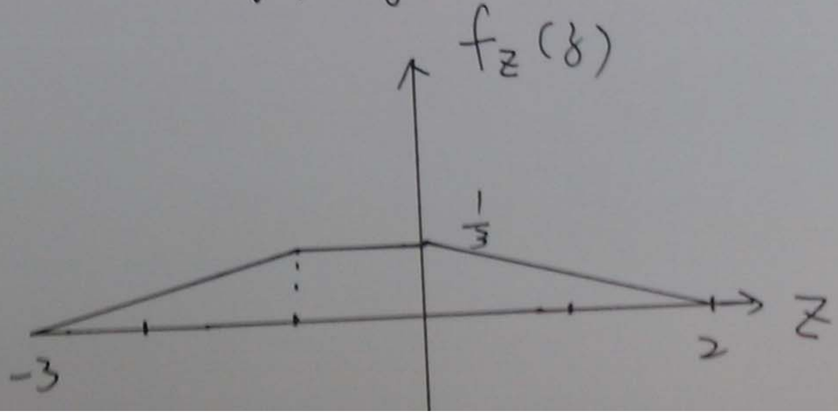
$F_Z(z) = \frac{1}{6}\left[\frac{[(1+z)+(3+z)] \times 2}{2}\right] = \frac{2+z}{3}$

$\Rightarrow f_Z(z) = \frac{1}{3}$

③ $-3 \le z \le -1$

$F_Z(z) = \frac{1}{6}\left[\frac{1}{2}(3+z)^2\right]$

$\Rightarrow f_Z(z) = \frac{3+z}{6}$



Sums of Independent Random Variables (1/2)

- We also can use the **convolution** method to obtain the distribution of $W = X + Y$
  - If X and Y are independent **discrete random variables** with integer values

$$p_W(w) = \mathbf{P}(X + Y = w) = \sum_{\{(x,y) \mid x+y=w\}} \mathbf{P}(X = x, Y = y)$$

$$= \sum_x \mathbf{P}(X = x, Y = w - x) = \sum_x \mathbf{P}(X = x)\mathbf{P}(Y = w - x)$$

$$= \sum_x p_X(x) p_Y(w - x) \quad \left(\text{also equivalent to } \sum_y p_X(w - y) p_Y(y) \right)$$

Convolution of PMFs of X and Y

Sums of Independent Random Variables (2/2)

– If X and Y are independent **continuous random variables**, the PDF $f_W(w)$ of $W = X + Y$ can be obtained by

$$f_W(w) = \int_{-\infty}^{\infty} f_X(x) f_Y(w-x)\,dx$$

Convolution of PMFs of X and Y

$$\left(\text{also equivalent to } \int_{-\infty}^{\infty} f_X(w-x) f_Y(y)\,dy\right)$$

Note that

$$\mathbf{P}(W \le w \mid X = x) = \mathbf{P}(X + Y \le w \mid X = x)$$
$$= \mathbf{P}(x + Y \le w)$$
$$= \mathbf{P}(Y \le w - x)$$

independence assumption

$$\Rightarrow F_{W|X}(w \mid x) = F_Y(w - x)$$

Differentiate the CDFs of both sides with respect to w

$$\Rightarrow f_{W|X}(w \mid x) = f_Y(w - x)$$

Applying the multiplication (chain) rule, we have

$$f_{W,X}(w,x) = f_X(x) f_{W|X}(w \mid x)$$
$$= f_X(x) f_Y(w - x)$$

Finally, by marginalization, we can have

$$f_W(w) = \int_{-\infty}^{\infty} f_{W,X}(w,x)\,dx$$
$$= \int_{-\infty}^{\infty} f_X(x) f_Y(w-x)\,dx$$

Illustrative Examples (1/4)

- **Example.** Let X and Y be independent and have PMFs given by

$$p_X(x)=\begin{cases}1/3, & \text{if } x=1,2,3,\\ 0, & \text{otherwise.}\end{cases} \qquad p_Y(y)=\begin{cases}1/2, & \text{if } y=0,\\ 1/3, & \text{if } y=1,\\ 1/6, & \text{if } y=2,\\ 0, & \text{otherwise.}\end{cases}$$

  - Calculate the PMF of $W=X+Y$ by convolution.

We know that the range of possible value of W are integers from the range [1, 5]

$$\begin{aligned}p_W(1)&=\sum_x p_X(x)p_Y(1-x)\\&=p_X(1)p_Y(0)\\&=1/3\cdot 1/2=1/6\end{aligned}$$

$$\begin{aligned}p_W(2)&=\sum_x p_X(x)p_Y(2-x)\\&=p_X(1)p_Y(1)+p_X(2)p_Y(0)\\&=1/3\cdot 1/3+1/3\cdot 1/2\\&=1/9+1/6=5/18\end{aligned}$$

$$\begin{aligned}p_W(3)&=\sum_x p_X(x)p_Y(3-x)\\&=p_X(1)p_Y(2)+p_X(2)p_Y(1)+p_X(3)p_Y(0)\\&=1/3\cdot 1/6+1/3\cdot 1/3+1/3\cdot 1/2\\&=1/18+1/9+1/6=1/3\end{aligned}$$

$$\begin{aligned}p_W(4)&=\sum_x p_X(x)p_Y(4-x)\\&=p_X(2)p_Y(2)+p_X(3)p_Y(1)\\&=1/3\cdot 1/6+1/3\cdot 1/3\\&=1/18+1/9=1/6\end{aligned}$$

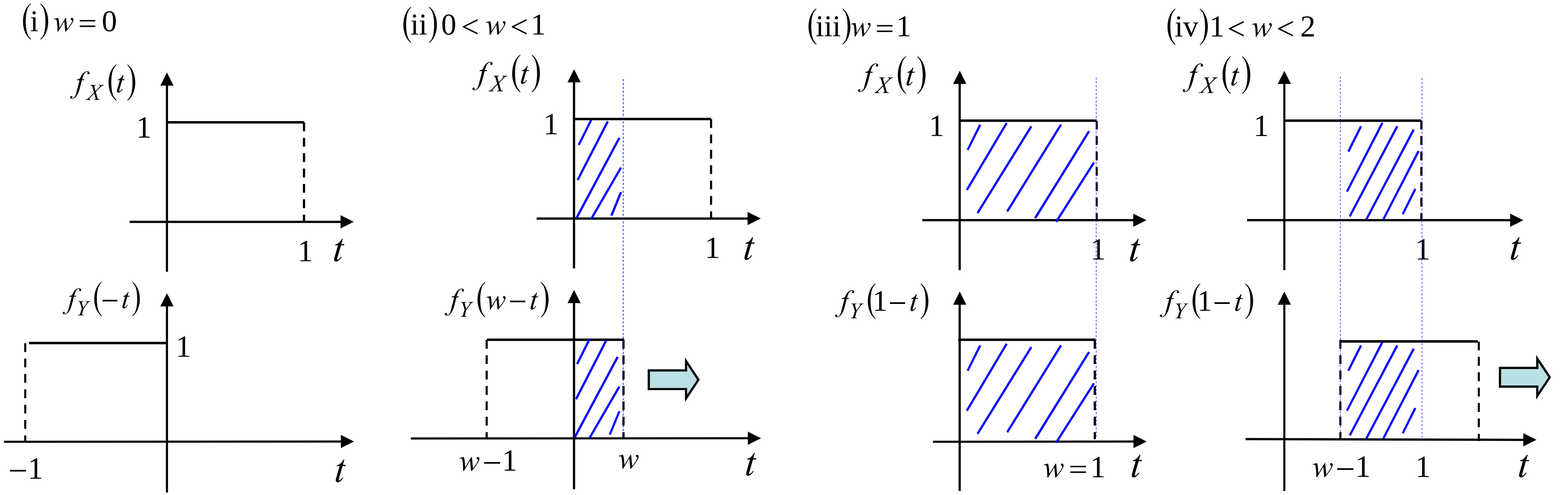
$$\begin{aligned}p_W(5)&=\sum_x p_X(x)p_Y(5-x)\\&=p_X(3)p_Y(2)\\&=1/3\cdot 1/6\\&=1/18\end{aligned}$$

Illustrative Examples (2/4)

- **Example 4.10.** The random variables X and Y are independent and uniformly distributed in the interval [0, 1]. The PDF of $W = X + Y$ is

$$f_W(w) = \int_{-\infty}^{\infty} f_X(t) f_Y(w-t)\,dt$$

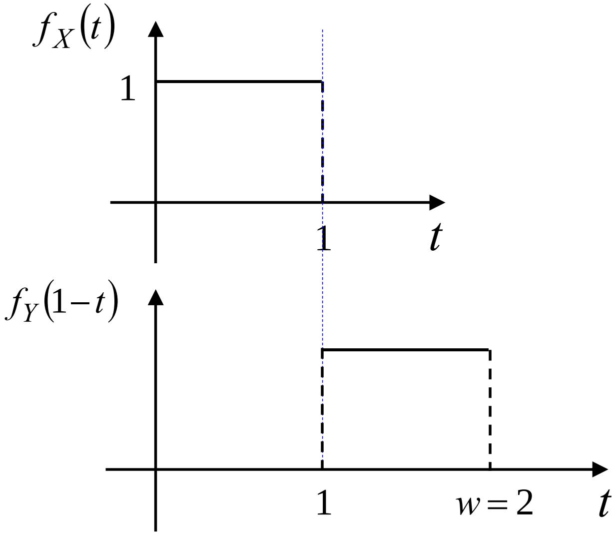
We know that the range of possible value of W are in the range [0, 2]

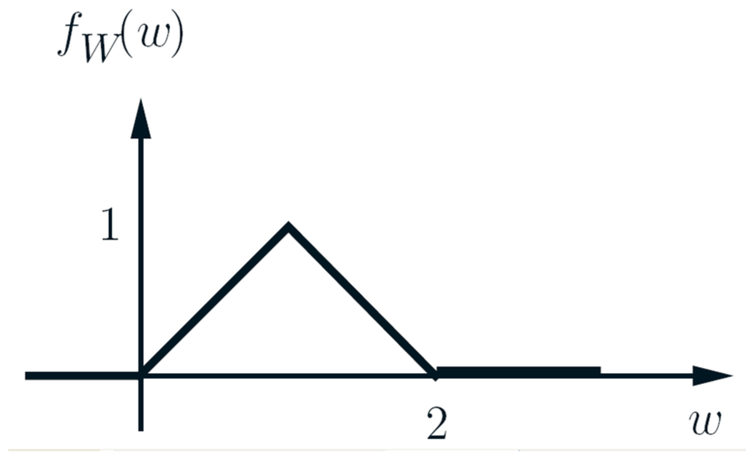


$f_W(0) = \int_0^0 f_X(t) f_Y(-t)\,dt = 0$ $\quad f_W(w) = \int_0^w f_X(t) f_Y(w-t)\,dt = w$ $\quad f_W(1) = \int_0^1 f_X(t) f_Y(1-t)\,dt = 1$ $\quad f_W(w) = \int_{w-1}^1 f_X(t) f_Y(1-t)\,dt = 2-w$

Illustrative Examples (3/4)

(iv) $w=2$





$$f_W(w)=\int_1^1 f_X(t) f_Y(1-t)\,dt=0$$

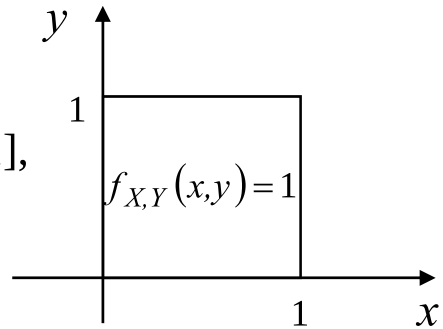
$$\therefore \quad f_W(w)=\begin{cases} w, & \text{if } 0\le w\le 1 \\ 2-w, & \text{if } 1\le w\le 2 \\ 0, & \text{otherwise} \end{cases}$$

$$\text{or as } f_W(w)=\begin{cases} \min\{1,w\}-\max\{0,\mathrm{w}-1\}, & 0\le w\le 2 \\ 0, & \text{otherwise} \end{cases}$$

shown in textbook

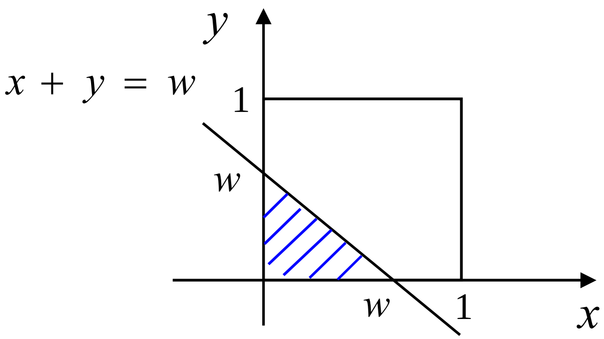
Illustrative Examples (4/4)

– Or, we can use the "**Derived Distribution**" method previously introduced



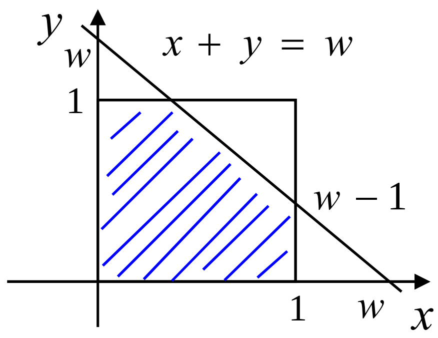
Since X and Y are indepent random varibles uniformly distribute d in [0, 1], we have their joint PDF $f_{X,Y}(x,y) = f_X(x) f_Y(y) = 1$, for $0 \le x,y \le 1$

(i) $0 \le w \le 1$



$$F_W(w) = \mathbf{P}(W \le w) = \mathbf{P}(X + Y \le w)$$
$$= \frac{1}{2} w^2$$
$$\Rightarrow f_W(w) = w$$

(ii) $1 \le w \le 2$

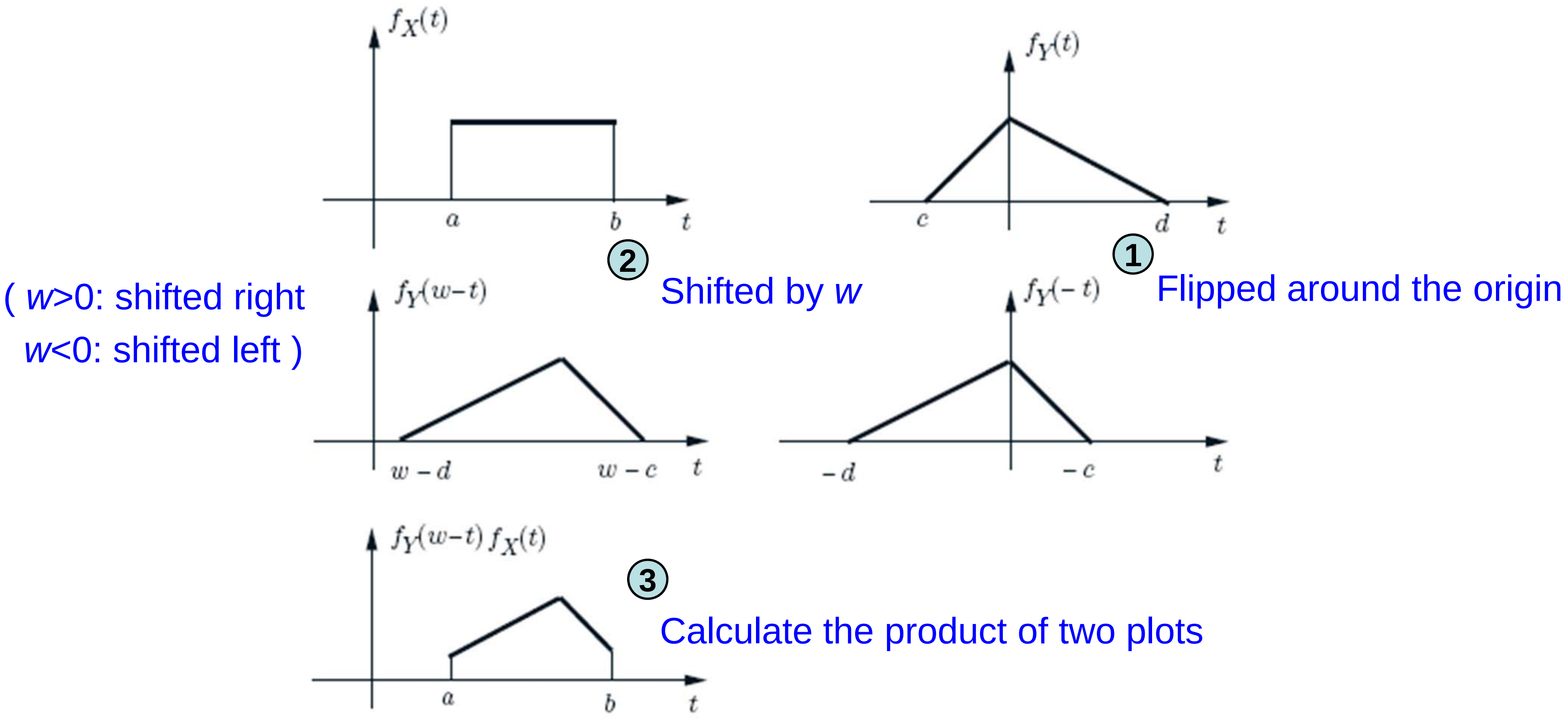


$$F_W(w) = \mathbf{P}(W \le w) = \mathbf{P}(X + Y \le w)$$
$$= 1 - \frac{1}{2}(2 - w)^2$$
$$\Rightarrow f_W(w) = 2 - w$$

$$\therefore f_W(w) = \begin{cases} w, & \text{if } 0 \le w \le 1 \\ 2 - w, & \text{if } 1 \le w \le 2 \\ 0, & \text{otherwise} \end{cases}$$

Graphical Calculation of Convolutions

- **Figure 4.10.** Illustration of the convolution calculation. For the value of W under consideration, $f_W(w)$ is equal to the integral of the function shown in the last plot.



Recitation

- SECTION 4.1 Derived Distributions
  - Problems 1, 4, 8, 11, 14