Continuous Random Variables: Joint PDFs, Conditioning, Expectation and Independence

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Reference:

- D. P. Bertsekas, J. N. Tsitsiklis, *Introduction to Probability* , Sections 3.4-3.6

Multiple Continuous Random Variables (1/2)

- Two continuous random variables X and Y associated with a common experiment are **jointly continuous** and can be described in terms of a **joint PDF** $f_{X,Y}$ satisfying

$$\mathbf{P}\big((X,Y)\in B\big)=\iint_{(x,y)\in B} f_{X,Y}(x,y)\,dx\,dy$$

  – $f_{X,Y}$ is a nonnegative function

  – Normalization Probability $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy = 1$

- Similarly, $f_{X,Y}(a,c)$ can be viewed as the "probability per unit area" in the vicinity of (a,c)

$$\mathbf{P}(a\le X\le a+\delta, c\le Y\le c+\delta)$$
$$=\int_a^{a+\delta}\int_c^{c+\delta} f_{X,Y}(x,y)\,dx\,dy = f_{X,Y}(a,c)\cdot\delta^2$$

  – Where δ is a small positive number

Multiple Continuous Random Variables (2/2)

- Marginal Probability

$$\mathbf{P}(X \in A) = \mathbf{P}(X \in A \text{ and } X \in (-\infty, \infty))$$
$$= \int_{X \in A} \int_{-\infty}^{\infty} f_{X,Y}(x, y) dy dx$$

  – We have already defined that

$$\mathbf{P}(X \in A) = \int_{X \in A} f_X(x) dx$$

    - We thus have the marginal PDF

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y) dy$$

Similarly

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x, y) dx$$

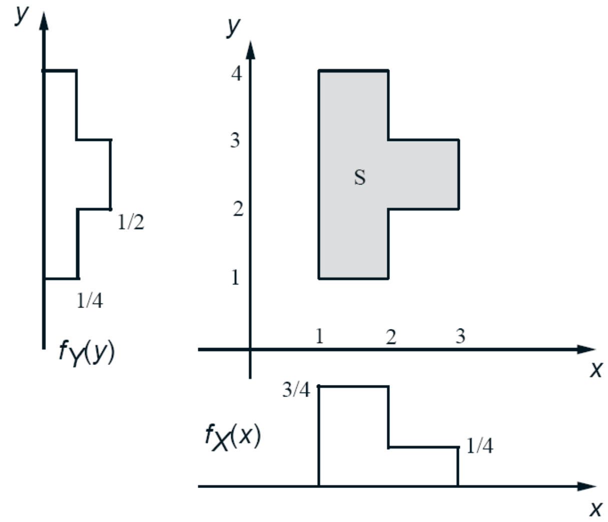
An Illustrative Example

- **Example 3.10. Two-Dimensional Uniform PDF**. We are told that the joint PDF of the random variables X and Y is a constant c on an area S and is zero outside. Find the value of c and the marginal PDFs of X and Y.

The correspond ing uniform joint PDF on an area S is defined to be (cf. Example 3.9)

$$f_{X,Y}(x,y) = \begin{cases} \dfrac{1}{\text{Size of area S}}, & \text{if } (x,y) \in S \\ 0, & \text{otherwise} \end{cases}$$

$$\Rightarrow f_{X,Y}(x,y) = \frac{1}{4} \quad \text{for} \quad (x,y) \in S$$



for $1 \le x \le 2$

$$\Rightarrow f_X(x) = \int_1^4 f_{X,Y}(x,y)dy = \int_1^4 \frac{1}{4} dy = \frac{3}{4}$$

for $2 \le x \le 3$

$$\Rightarrow f_X(x) = \int_2^3 f_{X,Y}(x,y)dy = \int_2^3 \frac{1}{4} dy = \frac{1}{4}$$

for $1 \le y \le 2$

$$\Rightarrow f_Y(y) = \int_1^2 f_{X,Y}(x,y)dx = \int_1^2 \frac{1}{4} dx = \frac{1}{4}$$

for $2 \le y \le 3$

$$\Rightarrow f_Y(y) = \int_1^3 f_{X,Y}(x,y)dx = \int_1^3 \frac{1}{4} dx = \frac{1}{2}$$

for $3 \le y \le 4$

$$\Rightarrow f_Y(y) = \int_1^2 f_{X,Y}(x,y)dx = \int_1^2 \frac{1}{4} dx = \frac{1}{4}$$

Joint CDFs

- If X and Y are two (either continuous or discrete) random variables associated with the same experiment , their joint cumulative distribution function (Joint CDF) is defined by

$$F_{X,Y}(x,y) = \mathbf{P}(X \le x, Y \le y)$$

  - If X and Y further have a joint PDF $f_{X,Y}$ (X and Y are continuous random variables) , then

$$F_{X,Y}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(s,t)\,ds\,dt$$

And

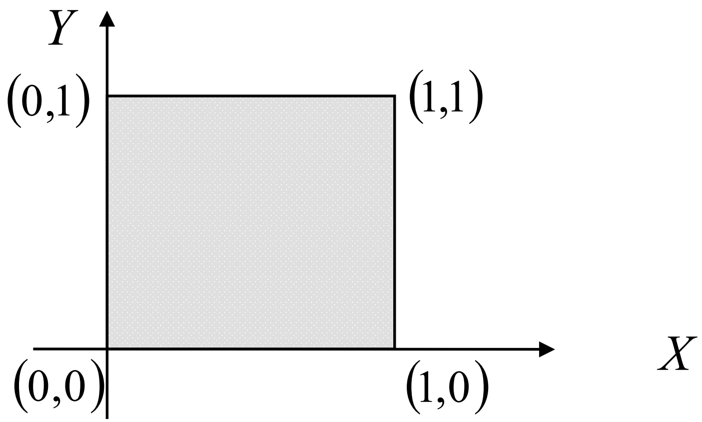
$$f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}(x,y)}{\partial x \partial y}$$

If $F_{X,Y}$ can be differentiated at the point (x,y)

An Illustrative Example

- **Example 3.12.** Verify that if *X* and *Y* are described by a uniform PDF on the unit square, then the joint CDF is given by

$$F_{X,Y}(x, y) = \mathbf{P}(X \le x, Y \le y) = xy, \quad \text{for } 0 \le x,y \le 1$$



$$\frac{\partial^2 F_{X,Y}(x, y)}{\partial x \partial y} = 1 = f_{X,Y}(x, y), \quad \text{for all } (x, y) \text{ in the unit square}$$

Expectation of a Function of Random Variables

- If X and Y are jointly continuous random variables, and g is some function, then $Z = g(X,Y)$ is also a random variable (can be continuous or discrete)
  - The expectation of Z can be calculated by

$$\mathbf{E}[Z] = \mathbf{E}[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y)\,dxdy$$

  - If Z is a linear function of X and Y, e.g., $Z = aX + bY$, then

$$\mathbf{E}[Z] = \mathbf{E}[aX + bY] = a\mathbf{E}[X] + b\mathbf{E}[Y]$$

    - Where a and b are scalars

We will see in Section 4.1 methods for computing the PDF of Z (if it has one).

More than Two Random Variables

- The joint PDF of three random variables X , Y and Z is defined in analogy with the case of two random variables

$$\mathbf{P}\left(\left(X, Y, Z\right) \in B\right) = \iiint_{\left(X, Y, Z\right) \in B} f_{X,Y,Z}\left(x, y, z\right) dxdydz$$

  – The corresponding marginal probabilities

$$f_{X,Y}\left(x, y\right) = \int_{-\infty}^{\infty} f_{X,Y,Z}\left(x, y, z\right) dz$$

$$f_X\left(x\right) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y,Z}\left(x, y, z\right) dydz$$

- The expected value rule takes the form

$$\mathbf{E}\left[g\left(X, Y, Z\right)\right] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g\left(x, y, z\right) f_{X,Y,Z}\left(x, y, z\right) dxdy\, dz$$

  – If g is linear (of the form $aX + bY + cZ$), then

$$\mathbf{E}\left[aX + bY + cZ\right] = a\mathbf{E}\left[X\right] + b\mathbf{E}\left[Y\right] + c\mathbf{E}\left[Z\right]$$

Conditioning PDF Given an Event (1/3)

- The conditional PDF of a continuous random variable X, given an event A
  - If A cannot be described in terms of X, the conditional PDF is defined as a nonnegative function $f_{X|A}(x)$ satisfying

$$\mathbf{P}(X \in B | A) = \int_B f_{X|A}(x) dx$$

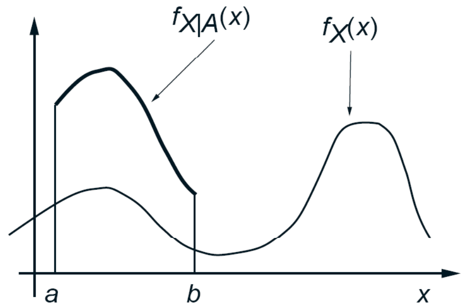
    - Normalization property

$$\int_{-\infty}^{\infty} f_{X|A}(x) dx = 1$$

Conditioning PDF Given an Event (2/3)

– If A can be described in terms of X (A is a subset of the real line with $\mathbf{P}(X \in A) > 0$), the conditional PDF is defined as a nonnegative function $f_{X|A}(x)$ satisfying

$$f_{X|A}(x) = \begin{cases} \dfrac{f_X(x)}{\mathbf{P}(X \in A)}, & \text{if } x \in A \\ 0, & \text{otherwise} \end{cases}$$



$f_{X|A}$ remains the same shape as f_X except that it is scaled along the vertical axis

- The conditional PDF is zero outside the conditioning event

and for any subset B

$$\mathbf{P}(X \in B | X \in A) = \frac{\mathbf{P}(X \in B, X \in A)}{\mathbf{P}(X \in A)} = \frac{\int_{A \cap B} f_X(x) dx}{\mathbf{P}(X \in A)} = \int_{A \cap B} f_{X|A}(x) dx$$

– Normalization Property $\int_{-\infty}^{\infty} f_{X|A}(x) dx = \int_A f_{X|A}(x) dx = 1$

Conditioning PDF Given an Event (3/3)

- If $A_1, A_2, \ldots, A_n$ are disjoint events with $\mathbf{P}(A_i) > 0$ for each i, that form a partition of the sample space, then

$$f_X(x) = \sum_{i=1}^{n} \mathbf{P}(A_i) f_{X|A_i}(x)$$

  – Verification of the above total probability theorem

think of $\{X \le x\}$ as an event B, and use the total probability theorem from Chapter 1

$$\mathbf{P}(X \le x) = \sum_{i=1}^{n} \mathbf{P}(A_i)\mathbf{P}(X \le x | A_i)$$

$$\Rightarrow \int_{-\infty}^{x} f_X(t)dt = \sum_{i=1}^{n} \mathbf{P}(A_i) \int_{-\infty}^{x} f_{X|A_i}(t)dt$$

Taking the derivative of both sides with respect to x

$$\Rightarrow f_X(x) = \sum_{i=1}^{n} \mathbf{P}(A_i) f_{X|A_i}(x)$$

Illustrative Examples (1/2)

- **Example 3.13. The exponential random variable is memoryless.**
  - The time T until a new light bulb burns out is exponential distribution. John turns the light on, leave the room, and when he returns, t time units later, find that the light bulb is still on, which corresponds to the event $A=\{T>t\}$
  - Let X be the additional time until the light bulb burns out. What is the conditional PDF of X given A ?

T is exponential

$$f_T(t)=\begin{cases}\lambda e^{-\lambda t}, & t>0\\ 0, & \text{otherwise}\end{cases}$$

$$P(T>t)=e^{-\lambda t}$$

$X=T-t,\ A=\{T>t\}$

The conditional CDF of X given A is defined by

$$P(X>x|A)=P(T-t>x|T>t)\ (\text{where } x\geq 0)$$

$$=P(T>t+x|T>t)=\frac{P(T>t+x \text{ and } T>t)}{P(T>t)}$$

$$=\frac{P(T>t+x)}{P(T>t)}$$

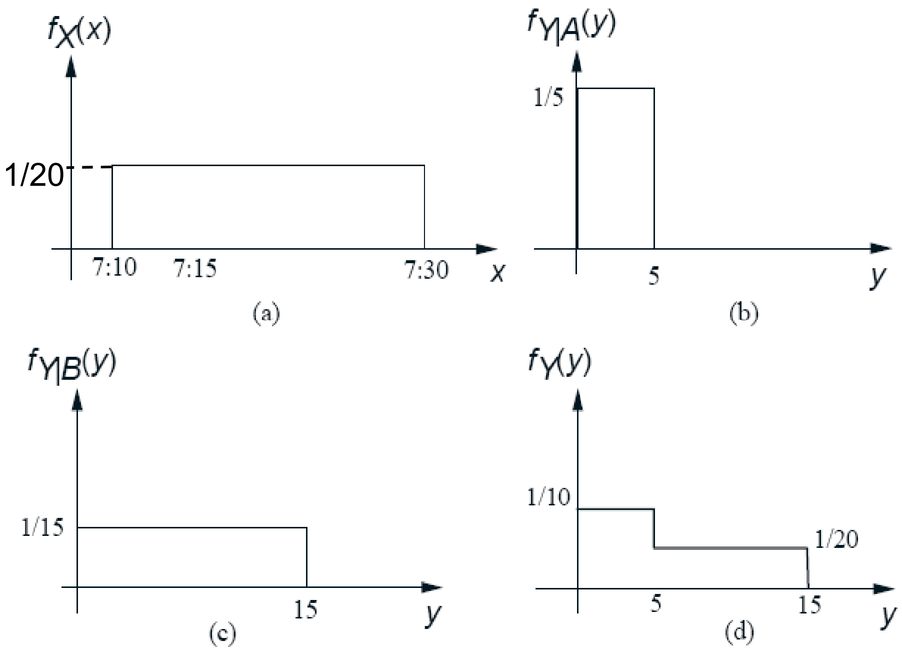
$$=\frac{e^{-\lambda(t+x)}}{e^{-\lambda t}}$$

$$=e^{-\lambda x}$$

$\therefore$ The conditional PDF of X given the event A is also exponential with parameter λ.

Illustrative Examples (2/2)

- **Example 3.14.** The metro train arrives at the station near your home every quarter hour starting at 6:00 AM. You walk into the station every morning between 7:10 and 7:30 AM, with the time in this interval being a uniform random variable. What is the PDF of the time you have to wait for the first train to arrive?



- The arrival time, denoted by X, is a uniform random variable over the interval 7:10 to 7:30
- Let random varible Y model the waiting time
- Let A be a event
  $A = \{7:10 \leq X \leq 7:15\}$ (You board the 7:15 train)
- Let B be a event
  $B = \{7:15 < X \leq 7:30\}$ (You board the 7:30 train)
- Let Y be uniform conditioned on A
- Let Y be uniform conditioned on B

Total Probability theorem:

$$P_Y(y) = P(A)P_{Y|A}(y) + P(B)P_{Y|B}(y)$$

$$\text{For } 0 \leq y \leq 5,\ P_Y(y) = \frac{1}{4}\cdot\frac{1}{5} + \frac{3}{4}\cdot\frac{1}{15} = \frac{1}{10}$$

$$\text{For } 5 < y \leq 15,\ P_Y(y) = \frac{1}{4}\cdot 0 + \frac{3}{4}\cdot\frac{1}{15} = \frac{1}{20}$$

Conditioning one Random Variable on Another

- Two continuous random variables X and Y have a joint PDF. For any y with $f_Y(y) > 0$, the conditional PDF of X given that $Y = y$ is defined by

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

  – Normalization Property $\int_{-\infty}^{\infty} f_{X|Y}(x|y)\,dx = 1$

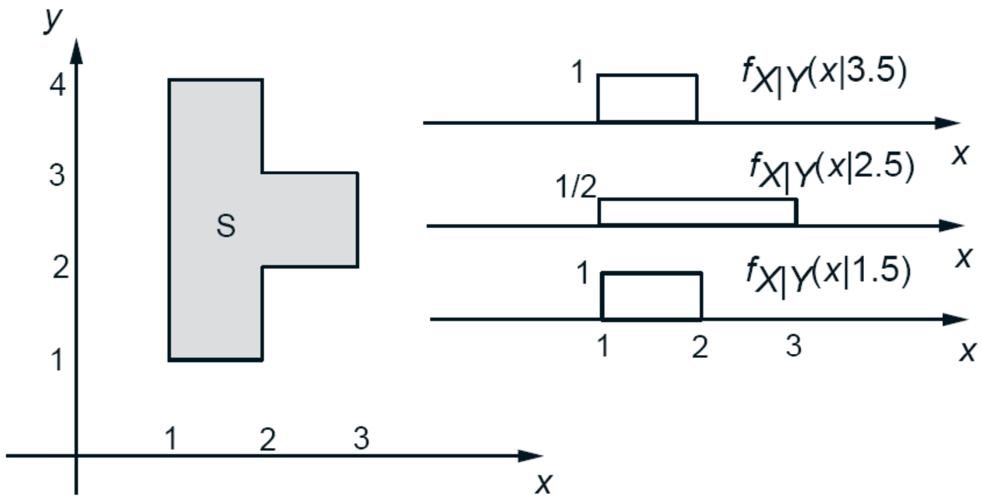
- The marginal, joint and conditional PDFs are related to each other by the following formulas

$$f_{X,Y}(x,y) = f_Y(y) f_{X|Y}(x|y),$$

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy.$$ marginalization

Illustrative Examples (1/2)

- Notice that the conditional PDF $f_{X|Y}(x|y)$ has the same shape as the joint PDF $f_{X,Y}(x,y)$, because the normalizing factor $f_Y(y)$ does not depend on x



$$f_{X|Y}(x|3.5) = \frac{f_{X,Y}(x,3.5)}{f_Y(3.5)} = \frac{1/4}{1/4} = 1$$

$$f_{X|Y}(x|3.5) = \frac{f_{X,Y}(x,2.5)}{f_Y(2.5)} = \frac{1/4}{1/2} = 1/2$$

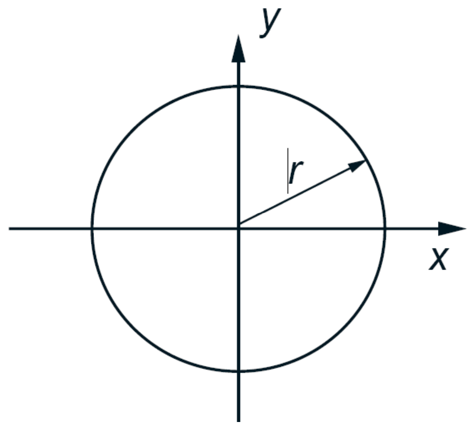
$$f_{X|Y}(x|3.5) = \frac{f_{X,Y}(x,1.5)}{f_Y(1.5)} = \frac{1/4}{1/4} = 1$$

cf. example 3.13

Figure 3.16: Visualization of the conditional PDF $f_{X|Y}(x|y)$.
Let X, Y have a joint PDF which is uniform on the set S. For each fixed y, we consider the joint PDF along the slice $Y = y$ and normalize it so that it integrates to 1

Illustrative Examples (2/2)

- **Example 3.15. Circular Uniform PDF.** Ben throws a dart at a circular target of radius r. We assume that he always hits the target, and that all points of impact (x, y) are equally likely, so that the joint PDF $f_{X,Y}(x,y)$ of the random variables x and y is uniform
  - What is the marginal PDF $f_Y(y)$



$$f_{X,Y}(x,y) = \begin{cases} \dfrac{1}{\text{area of the circle}}, & \text{if } (x,y) \text{ is in the circle} \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} \dfrac{1}{\pi r^2}, & x^2 + y^2 \le r^2 \\ 0, & \text{otherwise} \end{cases}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx = \int_{x^2+y^2 \le r^2} \frac{1}{\pi r^2}\,dx$$

$$= \frac{1}{\pi r^2} \int_{x^2+y^2 \le r^2} 1\,dx = \frac{1}{\pi r^2} \int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}} 1\,dx$$

$$= \frac{2}{\pi r^2} \sqrt{r^2 - y^2}, \ if\ |y| \le r$$

(Notice here that PDF $f_Y(y)$ is not uniform)

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

$$= \frac{\dfrac{1}{\pi r^2}}{\dfrac{2}{\pi r^2}\sqrt{r^2 - y^2}}$$

$$= \frac{1}{2\sqrt{r^2 - y^2}}, \qquad if\ x^2 + y^2 \le r^2$$

For each value y , $f_{X|Y}(x|y)$ is uniform

Conditional Expectation Given an Event

- The conditional expectation of a continuous random variable X, given an event A ($\mathbf{P}(A) > 0$), is defined by

$$\mathbf{E}\left[X \mid A\right] = \int_{-\infty}^{\infty} x f_{X \mid A}(x) dx$$

  – The conditional expectation of a function $g(X)$ also has the form

$$\mathbf{E}\left[g(X) \middle| A\right] = \int_{-\infty}^{\infty} g(x) f_{X|A}(x) dx$$

  – Total Expectation Theorem

$$\mathbf{E}[X] = \sum_{i=1}^{n} \mathbf{P}(A_i) \mathbf{E}\left[X \middle| A_i\right]$$

  and

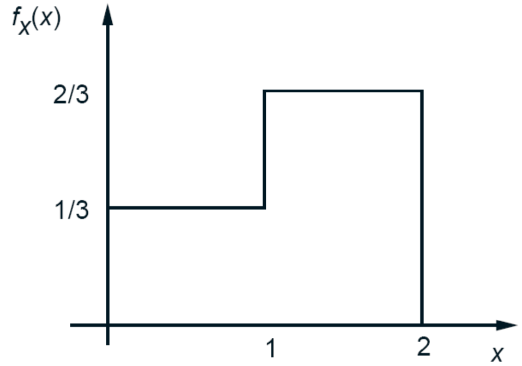
$$\mathbf{E}[g(X)] = \sum_{i=1}^{n} \mathbf{P}(A_i) \mathbf{E}\left[g(X) \middle| A_i\right]$$

    - Where $A_1, A_2, \ldots, A_n$ are disjoint events with $\mathbf{P}(A_i) > 0$ for each i, that form a partition of the sample space

An Illustrative Example

- **Example 3.17. Mean and Variance of a Piecewise Constant PDF.** Suppose that the random variable X has the piecewise constant PDF

$$f_X(x) = \begin{cases} 1/3, & \text{if } 0 \le x \le 1, \\ 2/3, & \text{if } 1 \le x \le 2, \\ 0, & \text{otherwise.} \end{cases}$$



Define event $A_1 = \{X \text{ lies in the first interval } [0,1]\}$

event $A_2 = \{X \text{ lies in the second interval } [1,2]\}$

$$\Rightarrow \mathbf{P}(A_1) = \int_0^1 1/3\,dx = 1/3,\ \mathbf{P}(A_2) = \int_1^2 2/3\,dx = 2/3$$

$$f_{X|A_1}(x) = \begin{cases} \dfrac{f_X(x)}{P(X \in A_1)} = 1, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases} \qquad f_{X|A_2}(x) = \begin{cases} \dfrac{f_X(x)}{P(X \in A_2)} = 1, & 1 \le x \le 2 \\ 0, & \text{otherwise} \end{cases}$$

> Recall that the mean and second moment of a uniform random variable over an interval $[a, b]$ is $(a+b)/2$ and $(a^2 + ab + b^2)/3$

$$\Rightarrow \mathrm{E}[X|A_1] = 1/2, \mathrm{E}[X^2|A_1] = 1/3$$

$$\mathrm{E}[X|A_2] = 3/2, \mathrm{E}[X^2|A_2] = 7/3$$

$$\Rightarrow \mathrm{E}[X] = \mathbf{P}(A_1)\mathrm{E}[X|A_1] + \mathbf{P}(A_2)\mathrm{E}[X|A_2]$$

$$= 1/3 \cdot 1/2 + 2/3 \cdot 3/2 = 7/6$$

$$\mathrm{E}[X^2] = \mathbf{P}(A_1)\mathrm{E}[X^2|A_1] + \mathbf{P}(A_2)\mathrm{E}[X^2|A_2]$$

$$= 1/3 \cdot 1/3 + 2/3 \cdot 7/3 = 15/9$$

$$\therefore \operatorname{var}(X) = 15/9 - (7/6)^2 = 11/36$$

Conditional Expectation Given a Random Variable

- The properties of unconditional expectation carry though, with the obvious modifications, to conditional expectation

$$\mathbf{E}\left[X \middle| Y = y\right] = \int_{-\infty}^{\infty} x f_{X|Y}\left(x|y\right) dx$$

$$\mathbf{E}\left[g(X) \middle| Y = y\right] = \int_{-\infty}^{\infty} g(x) f_{X|Y}\left(x|y\right) dx$$

$$\mathbf{E}\left[g(X, Y) \middle| Y = y\right] = \int_{-\infty}^{\infty} g(x, y) f_{X|Y}\left(x|y\right) dx$$

Total Probability/Expectation Theorems

- Total Probability Theorem
  - For any event A and a continuous random variable Y

$$\mathbf{P}(A) = \int_{-\infty}^{\infty} \mathbf{P}(A \mid Y = y) f_Y(y) dy$$

- Total Expectation Theorem
  - For any continuous random variables X and Y

$$\mathbf{E}[X] = \int_{-\infty}^{\infty} \mathbf{E}[X \mid Y = y] f_Y(y) dy$$

$$\mathbf{E}[g(X)] = \int_{-\infty}^{\infty} \mathbf{E}[g(X) \mid Y = y] f_Y(y) dy$$

$$\mathbf{E}[g(X,Y)] = \int_{-\infty}^{\infty} \mathbf{E}[g(X,Y) \mid Y = y] f_Y(y) dy$$

Independence

- Two continuous random variables X and Y are independent if

$$f_{X,Y}(x,y) = f_X(x) f_Y(y), \quad \text{for all } x,y$$

  - Since that

$$f_{X,Y}(x,y) = f_Y(y) f_{X|Y}(x|y) = f_X(x) f_{Y|X}(y|x)$$

    - We therefore have

$$f_{X|Y}(x|y) = f_X(x), \quad \text{for all } x \text{ and all } y \text{ with } f_Y(y) > 0$$

    - Or

$$f_{Y|X}(y|x) = f_Y(y), \quad \text{for all } y \text{ and all } x \text{ with } f_X(x) > 0$$

More Factors about Independence (1/2)

- If two continuous random variables X and Y are independent, then
  - Any two events of the forms $\{X \in A\}$ and $\{Y \in B\}$ are independent

$$\begin{aligned}\mathbf{P}(X \in A, Y \in B) &= \int_{x \in A} \int_{y \in B} f_{X,Y}(x, y) dy dx \\ &= \int_{x \in A} \int_{y \in B} f_X(x) f_Y(y) dy dx \\ &= \left[\int_{x \in A} f_X(x) dx\right]\left[\int_{y \in B} f_Y(y) dy\right] \\ &= \mathbf{P}(X \in A)\mathbf{P}(Y \in B)\end{aligned}$$

  - It also implies that

$$F_{X,Y}(x, y) = \mathbf{P}(X \le x, Y \le y) = \mathbf{P}(X \le x)\mathbf{P}(Y \le y) = F_X(x) F_Y(x)$$

  - The converse statement is also true (See the end-of-chapter problem 28)

More Factors about Independence (2/2)

- If two continuous random variables X and Y are independent, then
  - $\mathbf{E}[XY] = \mathbf{E}[X]\mathbf{E}[Y]$
  - $\mathrm{var}(X+Y) = \mathrm{var}(X) + \mathrm{var}(Y)$
  - The random variables $g(X)$ and $h(Y)$ are independent for any functions g and h
    - Therefore,

$$\mathbf{E}[g(X)h(Y)] = \mathbf{E}[g(X)]\mathbf{E}[h(Y)]$$

Recall: the Discrete Bayes' Rule

- Let $A_1, A_2, \ldots, A_n$ be disjoint events that form a partition of the sample space, and assume that $\mathbf{P}(A_i) \geq 0$, for all i . Then, for any event B such that $\mathbf{P}(B) > 0$ we have

$$\mathbf{P}(A_i|B) = \frac{\mathbf{P}(A_i)\mathbf{P}(B|A_i)}{\mathbf{P}(B)}$$

$$= \frac{\mathbf{P}(A_i)\mathbf{P}(B|A_i)}{\sum_{k=1}^{n} \mathbf{P}(A_k)\mathbf{P}(B|A_k)}$$

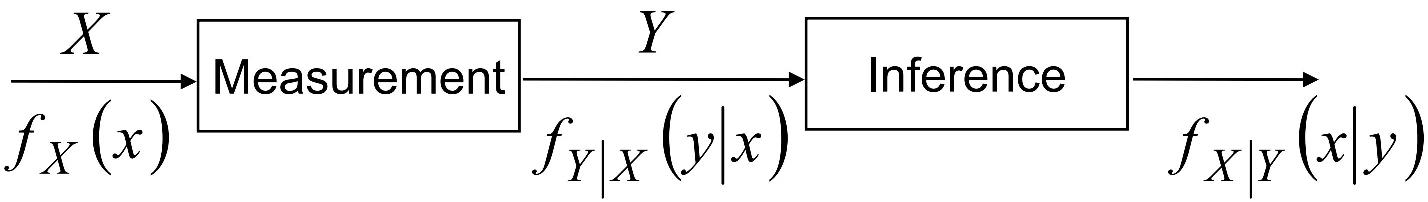
$$= \frac{\mathbf{P}(A_i)\mathbf{P}(B|A_i)}{\mathbf{P}(A_1)\mathbf{P}(B|A_1) + \cdots + \mathbf{P}(A_n)\mathbf{P}(B|A_n)}$$

Multiplication rule

Total probability theorem

Inference and the Continuous Bayes' Rule

- As we have a model of an underlying but unobserved phenomenon, represented by a random variable X with PDF f_X, and we make a noisy measurement Y, which is modeled in terms of a conditional PDF $f_{Y|X}$. Once the experimental value of Y is measured, what information does this provide on the unknown value of X?



$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{f_X(x) f_{Y|X}(y|x)}{\int_{-\infty}^{\infty} f_X(t) f_{Y|X}(y|t)\, dt}$$

Note that

$$f_X f_{Y|X} = f_{X,Y} = f_Y f_{X|Y}$$

Inference and the Continuous Bayes' Rule (2/2)

Inference about a Discrete Random Variable

- If the unobserved phenomenon is inherently discrete
  - Let N is a discrete random variable of the form $\{N = n\}$ that represents the different discrete probabilities for the unobserved phenomenon of interest, and p_N be the PMF of N

$$
\begin{aligned}
\mathbf{P}(N = n \mid Y = y) &\approx \mathbf{P}(N = n \mid y \le Y \le y + \delta) \\
&= \frac{\mathbf{P}(N = n)\mathbf{P}(y \le Y \le y + \delta \mid N = n)}{\mathbf{P}(y \le Y \le y + \delta)} \\
&\approx \frac{p_N(n) f_{Y|N}(y|n)\cancel{\delta}}{f_Y(y)\cancel{\delta}} \\
&= \frac{p_N(n) f_{Y|N}(y|n)}{\sum_i p_N(i) f_{Y|N}(y|i)}
\end{aligned}
$$

Total probability theorem

Illustrative Examples (1/2)

- **Example 3.19.** A lightbulb produced by the General Illumination Company is known to have an exponentially distributed lifetime Y. However, the company has been experiencing quality control problems. On any given day, the parameter $\Lambda = \lambda$ of the PDF of Y is actually a random variable, uniformly distributed in the interval $[1,\ 3/2]$.
  - If we test a lightbulb and record its lifetime ($Y = y$), what can we say about the underlying parameter λ ?

$$f_{Y|\Lambda}(y|\lambda) = \lambda e^{-\lambda y}, \quad y \geq 0, \lambda > 0$$

Conditioned on $\Lambda = \lambda$, Y has a exponential distribution with parameter λ

$$f_{\Lambda}(\lambda) = \begin{cases} 2, & \text{for } 1 \leq \lambda \leq 3/2 \\ 0, & \text{otherwise} \end{cases}$$

$$f_{\Lambda|Y}(\lambda|y) = \frac{f_{\Lambda}(\lambda) f_{Y|\Lambda}(y|\lambda)}{\int_1^{3/2} f_{\Lambda}(t) f_{Y|\Lambda}(y|t)\,dt} = \frac{2\lambda e^{-\lambda y}}{\int_1^{3/2} 2te^{-ty}\,dt}, \quad \text{for } 1 \leq \lambda \leq 3/2$$

Illustrative Examples (2/2)

- **Example 3.20. Signal Detection.** A binary signal S is transmitted, and we are given that $\mathbf{P}(S=1)=p$ and $\mathbf{P}(S=-1)=1-p$.
  - The received signal is $Y = S + N$, where N normal noise with zero mean and unit variance , independent of S .
  - What is the probability that $S=1$, as a function of the observed value y of Y ?

$$f_{Y|S}(y|s)=\frac{1}{\sqrt{2\pi}\sigma}e^{-(y-s)^2/2}, \text{for } s=1 \text{ and -1, and } -\infty \le y \le \infty$$

Conditioned on $S=s$, Y has a normal distribution with mean s and unit variance

$$\mathbf{P}(S=1|Y=y)=\frac{p_S(1)f_{Y|S}(y|1)}{f_Y(y)}=\frac{p_S(1)f_{Y|S}(y|1)}{p_S(1)f_{Y|S}(y|1)+p_S(-1)f_{Y|S}(y|-1)}$$

$$=\frac{p\cancel{\frac{1}{\sqrt{2\pi}}}e^{-(y-1)^2/2}}{p\cancel{\frac{1}{\sqrt{2\pi}}}e^{-(y-1)^2/2}+(1-p)\cancel{\frac{1}{\sqrt{2\pi}}}e^{-(y+1)^2/2}}$$

$$=\frac{\cancel{e^{-(y^2+1)/2}}\cdot pe^{y}}{\cancel{e^{-(y^2+1)/2}}\cdot pe^{y}+\cancel{e^{-(y^2+1)/2}}\cdot(1-p)e^{-y}}=\frac{pe^{y}}{pe^{y}+(1-p)e^{-y}}$$

Inference Based on a Discrete Random Variable

- The earlier formula expressing $\mathbf{P}(A|Y=y)$ in terms of $f_{Y|A}(y)$ can be turned around to yield

$$f_{Y|A}(y) = \frac{f_Y(y)\mathbf{P}(A|Y=y)}{\mathbf{P}(A)}$$

$$= \frac{f_Y(y)\mathbf{P}(A|Y=y)}{\int_{-\infty}^{\infty} f_Y(t)\mathbf{P}(A|Y=t)dt}$$

?

$$\mathbf{P}(A)f_{Y|A}(y) = f_Y(y)\mathbf{P}(A|Y=y)$$

$$\Rightarrow \int_{-\infty}^{\infty} \mathbf{P}(A)f_{Y|A}(y)dy = \int_{-\infty}^{\infty} f_Y(y)\mathbf{P}(A|Y=y)dy$$

$$\Rightarrow \mathbf{P}(A) = \int_{-\infty}^{\infty} f_Y(y)\mathbf{P}(A|Y=y)dy \ (\because \text{normalizat ion property } : \int_{-\infty}^{\infty} f_{Y|A}(y)dy = 1)$$

Recitation

- SECTION 3.4 Joint PDFs of Multiple Random Variables
  - Problems 15, 16
- SECTION 3.5 Conditioning
  - Problems 18, 20, 23, 24
- SECTION 3.6 The Continuous Bayes' Rule
  - Problems 34, 35