Further Topics on Random Variables: Others

1. Sum of a Random Number of Independent Random Variables
2. Covariance and Correlation

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Reference:
- D. P. Bertsekas, J. N. Tsitsiklis, *Introduction to Probability* , Sections 4.2-3

Sum of a Random Number of Independent Random Variables (1/4)

$$\underbrace{X_1, X_2, X_3, \ldots, X_N}_{Y = X_1 + X_2 + \cdots + X_N}, \ldots \ldots \ldots$$

- If we know that
  - N is a random variable taking positive integer values $N = 1, 2, \ldots$
  - $X_1, X_2, \ldots$ are independent, identically distributed (i.i.d.) random variables (with common mean μ and variance σ^2)
    - A subset of X_i's ($X_1, X_2, \cdots, X_N$) are independent as well
- What are the formulas for the mean, variance, and the transform of Y ?

$$Y = X_1 + X_2 + \cdots + X_N$$

Sum of a Random Number of Independent Random Variables (2/4)

- If we fix some number n, the random variable $X_1 + X_2 + \cdots + X_n$ is independent of random variable N

$$\begin{aligned} &\mathbf{E}[Y \mid N = n] \\ &= \mathbf{E}[X_1 + X_2 + \cdots + X_N \mid N = n] \\ &= \mathbf{E}[X_1 + X_2 + \cdots + X_n \mid N = n] \\ &= \mathbf{E}[X_1 + X_2 + \cdots + X_n] \\ &= n\mathbf{E}[X_i] = n\mu \end{aligned}$$

- $\mathbf{E}[Y \mid N]$ can be viewed as a function of random variable N
  - $\mathbf{E}[Y \mid N]$ is a random variable
  - The mean of $\mathbf{E}[Y \mid N]$ (i.e. $\mathbf{E}[Y]$) can be calculated by using the law of iterated expectations

$$\mathbf{E}[Y] = \mathbf{E}[\mathbf{E}[Y \mid N]] = \mathbf{E}[N\mu] = \mu\mathbf{E}[N]$$

Sum of a Random Number of Independent Random Variables (3/4)

- Similarly, $\mathrm{var}(Y|N=n)$ can be expressed as

$$
\begin{aligned}
&\mathrm{var}(Y|N=n) \\
&= \mathrm{var}(X_1 + X_2 + \cdots + X_N | N = n) \\
&= \mathrm{var}(X_1 + X_2 + \cdots + X_n | N = n) \\
&= \mathrm{var}(X_1 + X_2 + \cdots + X_n) \\
&= n\sigma^2
\end{aligned}
$$

  - $\mathrm{var}(Y|N)$ can be viewed as a function of random variable N
    - $\mathrm{var}(Y|N)$ is a random variable
  - The variance of Y can be calculated using the law of total variance

$$
\begin{aligned}
\mathrm{var}(Y) &= \mathbf{E}[\mathrm{var}(Y|N)] + \mathrm{var}(\mathbf{E}[Y|N]) \\
&= \mathbf{E}[N\sigma^2] + \mathrm{var}(N\mu) \\
&= \sigma^2 \mathbf{E}[N] + \mu^2 \mathrm{var}(N)
\end{aligned}
$$

Sum of a Random Number of Independent Random Variables (4/4)

- Similarly, $\mathbf{E}\left[e^{sY} \mid N=n\right]$ can be expressed as

$$
\begin{aligned}
&\mathbf{E}\left[e^{sY} \mid N=n\right] \\
&= \mathbf{E}\left[e^{s(X_1+X_2+\cdots+X_N)} \mid N=n\right] = \mathbf{E}\left[e^{s(X_1+X_2+\cdots+X_n)} \mid N=n\right] \\
&= \mathbf{E}\left[e^{s(X_1+X_2+\cdots+X_n)}\right] = \mathbf{E}\left[e^{sX_1} e^{sX_2} \cdots e^{sX_n}\right] \\
&= \left(M_X(s)\right)^n
\end{aligned}
$$

  – $\mathbf{E}\left[e^{sY} \mid N\right]$ can be viewed as a function of random variable N
    - $\mathbf{E}\left[e^{sY} \mid N\right]$ is a random variable
    - The mean of $\mathbf{E}\left[e^{sY} \mid N\right]$ (i.e. the transform of Y, $\mathbf{E}\left[e^{sY}\right]$) can be calculated by using the law of iterated expectations

$$
M_Y(s) = \mathbf{E}\left[e^{sY}\right] = \mathbf{E}\left[\mathbf{E}\left[e^{sY} \mid N\right]\right] = \mathbf{E}\left[\left(M_X(s)\right)^N\right] = \sum_{n=1}^{\infty} \left(M_X(s)\right)^n p_N(n)
$$

Properties of the Sum of a Random Number of Independent Random Variables

Let $X_1, X_2, \ldots$ be random variables with common mean μ and common variance σ^2. Let N be a random variable that takes nonnegative integer values. We assume that all of these random variables are independent, and consider

$$Y = X_1 + \cdots + X_N.$$

Then,

- $\mathbf{E}[Y] = \mu \mathbf{E}[N]$. $\Rightarrow \mathbf{E}[Y] = \mathbf{E}[N]\mathbf{E}[X_i]$
- $\text{var}(Y) = \sigma^2 \mathbf{E}[N] + \mu^2 \text{var}(N)$. $\Rightarrow \text{var}(Y) = \mathbf{E}[N]\text{var}(X_i) + (\mathbf{E}[X_i])^2\,\text{var}(N)$
- The transform $M_Y(s)$ is found by starting with the transform $M_N(s)$ and replacing each occurrence of e^s with $M_X(s)$.

Illustrative Examples (1/5)

- **Example 4.21.** A remote village has three gas stations, and each one of them is open on any given day with probability 1/2, independently of the others. The amount of gas available in each gas station is unknown and is uniformly distributed between 0 and 1000 gallons.
  - We wish to characterize the distribution (Y) of the total amount of gas available at the gas stations that are open

$$Y = X_1 + \cdots$$

Total amount of gas available

The amount of gas provided by one gas station, out of three (X_i is uniformly distributed)

① The number N of gas stations open at a day is a binomial distribution with parameter $(3, p)$
$\Rightarrow$ the transform of random variable N is

$$M_N(s) = \left(1 - p + pe^s\right)^3 = \frac{1}{8}\left(1 + e^s\right)^3$$

② The transform of X_i (uniformlly distributed) is :

$$M_X(s) = \int_0^{1000} e^{sx} \cdot \frac{1}{1000} dx$$

$$= \frac{e^{1000s} - 1}{1000s}$$

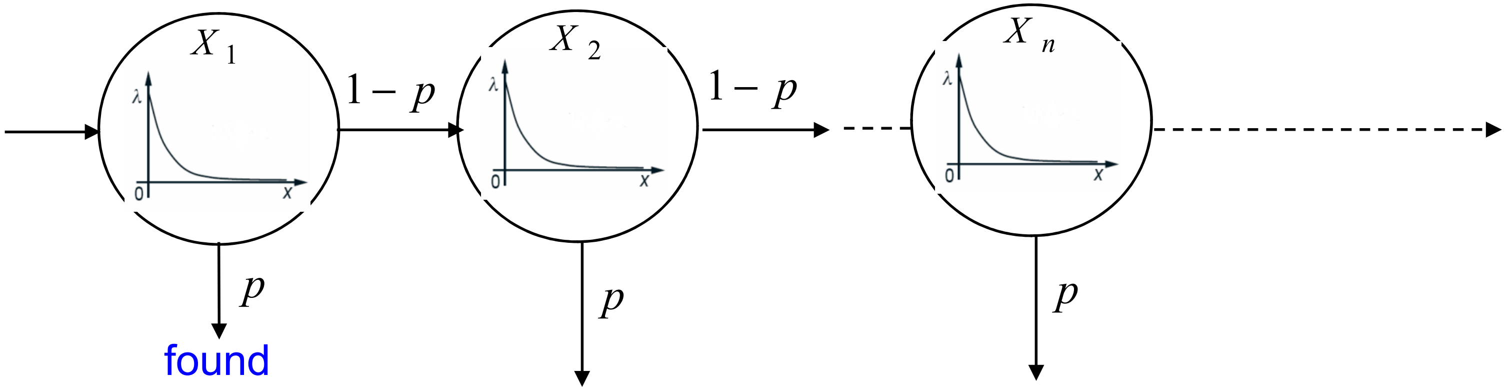
③ Using the property introduced in the previous slide, we have

$$M_Y(s) = \frac{1}{8}\left(1 + \left(\frac{e^{1000s} - 1}{1000s}\right)\right)^3$$

Illustrative Examples (2/5)

- **Example 4.22. Sum of a Geometric Number of Independent Exponential Random Variables.**
  - Jane visits a number of bookstores, looking for ***Great Expectations***. Any given bookstore carries the book with probability p, independently of the others. In a typical bookstore visited, Jane spends a random amount of time, exponentially distributed with parameter λ, until she either finds the book or she decides that the bookstore does not carry it. Assuming that Jane will keep visiting bookstores until she buys the book and that the time spent in each is independent of everything else
  - We wish to determine the mean, variance, and PDF of the total time spent in bookstores.

Illustrative Examples (3/5)



found

$Y = X_1$

Total amount of time spent

The amount of time spent in a given bookstore

found

$Y = X_1 + X_2$

found

$Y = X_1 + X_2 + \cdots + X_n$

①

$$\Rightarrow \mathbf{E}[Y] = \mathbf{E}[N]\mathbf{E}[X_i] = \frac{1}{p} \cdot \frac{1}{\lambda}$$

②

$$\begin{aligned} \operatorname{var}(Y) &= \mathbf{E}[N]\operatorname{var}(X_i) + (\mathbf{E}[X_i])^2 \operatorname{var}(N) \\ &= \frac{1}{p} \cdot \frac{1}{\lambda^2} + \left(\frac{1}{\lambda}\right)^2 \cdot \frac{1-p}{p^2} \\ &= \frac{1}{\lambda^2 p^2} \end{aligned}$$

Illustrative Examples (4/5)

③

$$M_X(s) = \frac{\lambda}{\lambda - s}$$

$$M_N(s) = \frac{pe^s}{1-(1-p)e^s}$$

$$\Rightarrow M_Y(s) = \frac{p\dfrac{\lambda}{\lambda - s}}{1-(1-p)\dfrac{\lambda}{\lambda - s}} = \frac{p\lambda}{\lambda - s - \lambda + p\lambda} = \frac{p\lambda}{p\lambda - s}$$

$\therefore$ Y is an exponentia lly distribute d random variable with parameter $p\lambda$

$$f_Y(y) = p\lambda e^{-p\lambda y}, \quad y \ge 0$$

Recall that if Y is the sum of a fixed number of independent random variables (e.g., $Y = X_1 + X_2$), its associated transform $M_Y(s)$ is (Assume that X_1, X_2 are identical exponential distributions with parameter λ)

$$M_Y(s) = \left(\frac{\lambda}{\lambda - s}\right)^2$$

$\Rightarrow$ Y is not an exponential random variable

Illustrative Examples (5/5)

- **Example 4.23. Sum of a Geometric Number of Independent Geometric Random Variables.**
  - This example is a discrete counterpart of the preceding one.
  - We let N be geometrically distributed with parameter p. We also let each random variable X_i be geometrically distributed with parameter q. We assume that all of these random variables are independent.

$$M_X(s) = \frac{qe^s}{1-(1-q)e^s}$$

$$M_N(s) = \frac{pe^s}{1-(1-p)e^s}$$

$$\Rightarrow M_Y(s) = \frac{p\dfrac{qe^s}{1-(1-q)e^s}}{1-(1-p)\dfrac{qe^s}{1-(1-q)e^s}} = \frac{pqe^s}{1-(1-q)e^s-(1-p)qe^s} = \frac{pqe^s}{1-(1-pq)e^s}$$

$\therefore$ Y is a geometric distributed random variable with parameter pq

Covariance (1/2)

- The covariance of two random variables X and Y is denoted by

$$\operatorname{cov}(X, Y) = \mathbf{E}[(X - \mathbf{E}[X])(Y - \mathbf{E}[Y])]$$

  – An alternative formula is

$$\operatorname{cov}(X, Y) = \mathbf{E}[XY] - \mathbf{E}[X]\mathbf{E}[Y]$$

- Note that if X and Y are independent

$$\mathbf{E}[XY] = \mathbf{E}[X]\mathbf{E}[Y]$$

  – Therefore

$$\operatorname{cov}(X, Y) = 0$$

- Note still that knowing $\operatorname{cov}(X, Y) = 0$ does not indicate that X and Y are independent !

Covariance (2/2)

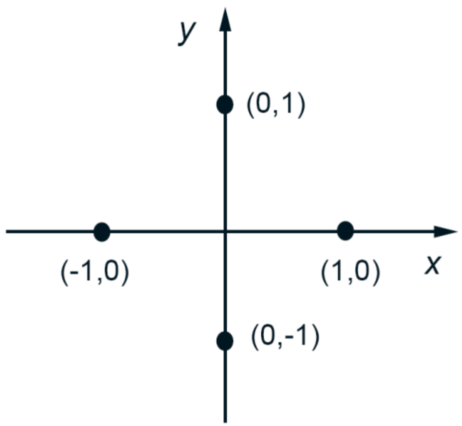
- **Example 4.24.** The pair of random variables (*X, Y*) takes the values (1, 0), (0, 1), (−1, 0), and (0,−1), each with probability ¼ Thus, the marginal pmfs of *X* and *Y* are symmetric around 0, and **E**[*X*] = **E**[*Y*] = 0
  – Furthermore, for all possible value pairs (*x, y*), either *x* or *y* is equal to 0, which implies that *XY* = 0 and **E**[*XY*] = 0. Therefore, cov(*X, Y*) = **E**[(*X* − **E**[*X*])(*Y* − **E**[*Y*])] = **E**[*XY*] = 0, and *X* and *Y* are uncorrelated
  – However, X and Y are not independent since, for example, a nonzero value of X fixes the value of Y to zero

$$P(X=0)=\frac{1}{2}$$

$$P(X=1)=P(X=-1)=\frac{1}{4}$$

$$P(Y=0)=\frac{1}{2}$$

$$P(Y=1)=P(Y=-1)=\frac{1}{4}$$



For example :

$$P(X=1,Y=1)=\frac{1}{4}$$

$$\neq P(X=1)P(Y=1)=\frac{1}{16}$$

Correlation (1/2)

- Also denoted as "Correlation Coefficient"
- The correlation coefficient of two random variables X and Y is defined as

$$\rho = \frac{\operatorname{cov}(X, Y)}{\sqrt{\operatorname{var}(X)\operatorname{var}(Y)}}$$

– It can be shown that

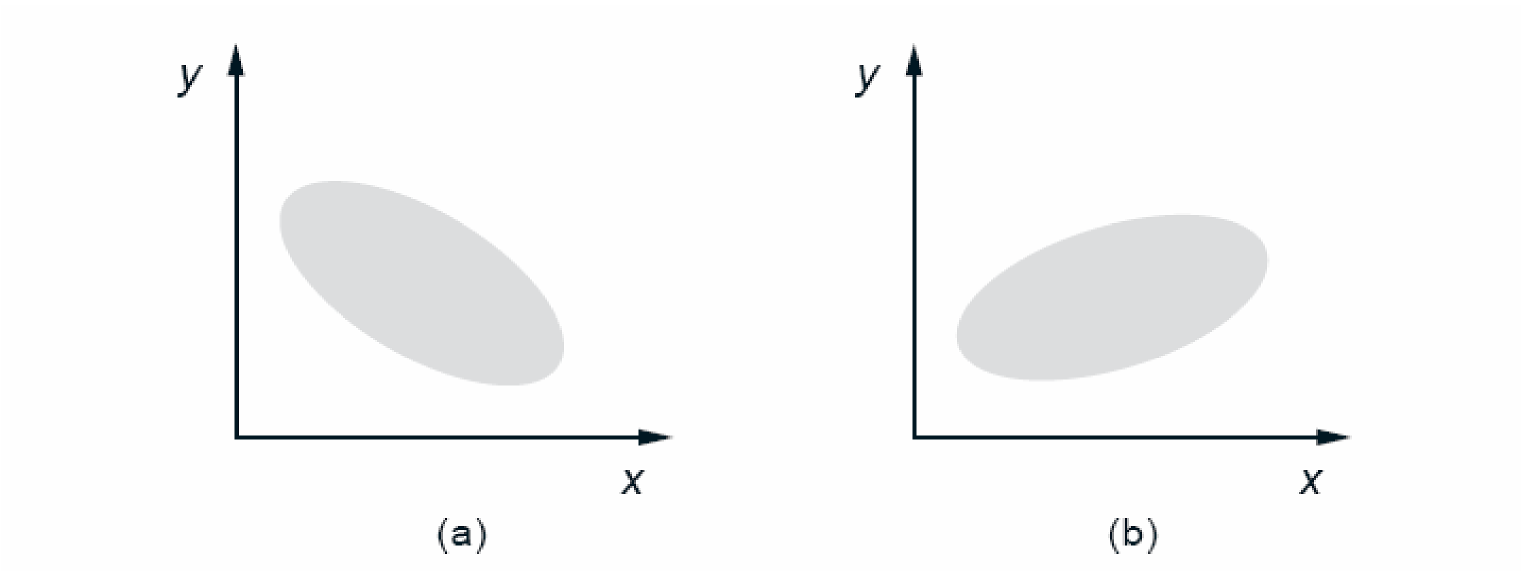
$$-1 \leq \rho \leq 1$$

Note that
the sign of ρ only depends on $\operatorname{cov}(X,Y)$

- $\rho > 0$: positively correlated
- $\rho < 0$: negatively correlated
- $\rho = 0$: uncorrelated $\left(\Rightarrow \operatorname{cov}(X,Y) = 0 \right)$

Correlation (2/2)

- **Figure 4.7:** Examples of positively (b) and negatively (c) correlated random variables



$\text{cov}(X,Y) < 0$

$\text{cov}(X,Y) > 0$