Linear Algebra
Quiz 4 (Brief Solution)

11:10 a.m. - 12:10 a.m., December 18, 2015

Note: You have to answer the questions with supporting explanations (i.e., show all your work) if needed.

1. Show that the three vectors $\mathbf{v}_1$=(0, 3, 1, -1), $\mathbf{v}_2$=(6, 0, 5, 1) and $\mathbf{v}_2$=(4, -7, 1, 3) form a **linearly dependent** set in R^4. (15%)

   Ans.:

   Let $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 = \mathbf{0}$

   It is easy to find (with elementary row operations) a set of not all zero values of k_1, k_2, k_3 (i.e., a nontrivial solution), for example $k_1 = 7_1, k_2 = -2, k_3 = 3$, such that $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 = \mathbf{0}$.

2. Find the coordinate vector of $\mathbf{p}=2-x-x^2$ relative to the following three basis vectors: $\mathbf{p}_1=1+x$, $\mathbf{p}_2=1+x^2$ and $\mathbf{p}_3=x+x^2$. (15%)

   Ans.:

   Let $c_1\mathbf{p}_1 + c_2\mathbf{p}_2 + c_3\mathbf{p}_3 = \mathbf{p}$. It is easy to find that $c_1 = 1_1, c_2 = 1, c_3 = -2$.

   Therefore, the coordinate vector of $\mathbf{p}$ relative to $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ is $[1, 1, -2]^T$

3. Given a matrix A shown below:

$$A = \begin{bmatrix} 1 & 1 & 2 & 4 & 2 \\ 2 & 1 & 2 & 3 & 1 \\ 1 & 2 & 4 & 9 & 2 \\ 6 & 3 & 6 & 9 & 3 \end{bmatrix}$$

   (i) Find a basis for the column space of A consisting entirely of column vectors of A. (10%)

   (ii) Find a basis for the row space of A consisting entirely of row vectors of A. (10%)

   (iii) Find a basis for the null space of A. (10%)

   (iv) What is the rank of A ? and what is the dimension of the null space of A? (10%)

   Ans.:

   (i) basis for the column space $= \{[1, 2, 1, 6]^T, [1, 1, 2, 3]^T, [2, 1, 2, 3]^T\}$

   (ii) basis for the null space $= \{[1, 1, 2, 4, 2], [2, 1, 2, 3, 1], [1, 2, 4, 9, 2]\}$

   (iii) basis for the null space $= \{[1, -5, 0, 1, 0], [0, -2, 1, 0, 0]\}$

   (iv) the rank of A = 3; the dimension of the null space of A = 2

4. Consider the bases $B=\{\mathbf{u}_1, \mathbf{u}_2\}$ and $B'=\{\mathbf{v}_1, \mathbf{v}_2\}$ for R^2, where

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

   (i) Find the transition (change-of-coordinates) matrix from B to B'. (10%)

   (ii) Find the transition (change-of-coordinates) matrix from B' to B. (10%)

   (iii) Given that $[\mathbf{w}]_B = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$, compute $[\mathbf{w}]_{B'}$. (10%)

Ans.:

(i) $P_{B\to B'} = \begin{bmatrix} -1 & -3 \\ 1 & 2 \end{bmatrix}$

(ii) $P_{B'\to B} = \begin{bmatrix} 2 & 3 \\ -1 & -1 \end{bmatrix}$

(iii) $[\mathbf{w}]_{B'} = \begin{bmatrix} -7 \\ 6 \end{bmatrix}$