Chapter 2

Determinants

Outline

- 2.1 Determinants by Cofactor Expansion
- 2.2 Evaluating Determinants by Row Reduction
- 2.3 Properties of Determinants; Cramer’s Rule

2.1

Determinants by Cofactor Expansion

Determinant

- Recall from Theorem 1.4.5 that the 2×2 matrix
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
is invertible if $ad - bc \neq 0$. It is called the ***determinant*** (行列式) of the matrix *A* and is denoted by the symbol det(*A*) or |*A*|
$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Minor and Cofactor

- Definition
  - Let A be $n \times n$
    - The (i,j)-minor (子行列式) of A, denoted M_{ij} is the determinant of the $(n-1) \times (n-1)$ matrix formed by deleting the ith row and jth column from A
    - The (i,j)-cofactor (餘因子) of A, denoted C_{ij}, is $(-1)^{i+j} M_{ij}$
- Remark
  - Note that $C_{ij} = \pm M_{ij}$ and the signs $(-1)^{i+j}$ in the definition of cofactor form a checkerboard pattern:

$$\begin{bmatrix} + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

Example

- Let
$$A = \begin{bmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{bmatrix}$$
- The minor of entry a_{11} is $M_{11} = \begin{vmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{vmatrix} = \begin{vmatrix} 5 & 6 \\ 4 & 8 \end{vmatrix} = 16$
- The cofactor of a_{11} is $C_{11} = (-1)^{1+1}M_{11} = M_{11} = 16$
- Similarly, the minor of entry a_{32} is $M_{32} = \begin{vmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{vmatrix} = \begin{vmatrix} 3 & -4 \\ 2 & 6 \end{vmatrix} = 26$
- The cofactor of a_{32} is $C_{32} = (-1)^{3+2}M_{32} = -M_{32} = -26$

Cofactor Expansion of a 2 x 2 Matrix

- For the matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

$$C_{11} = M_{11} = a_{22} \qquad C_{12} = -M_{12} = -a_{21}$$

$$C_{21} = -M_{21} = -a_{12} \qquad C_{22} = M_{22} = a_{11}$$

$$det(A) = a_{11}a_{22} - a_{12}a_{21}$$

$$\begin{aligned} \det(A) &= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \\ &= a_{11}C_{11} + a_{12}C_{12} \\ &= a_{21}C_{21} + a_{22}C_{22} \\ &= a_{11}C_{11} + a_{21}C_{21} \\ &= a_{12}C_{12} + a_{22}C_{22} \end{aligned}$$

These are called cofactor expansions of *A*

Cofactor Expansion

- Theorem 2.1.1 (Expansions by Cofactors)
  - The determinant of an $n \times n$ matrix A can be computed by multiplying the entries in any row (or column) by their cofactors and adding the resulting products; that is, for each $1 \le i, j \le n$

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \ldots + a_{nj}C_{nj}$$

(cofactor expansion along the jth column)

and

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \ldots + a_{in}C_{in}$$

(cofactor expansion along the ith row)

- Example

$$\det(A) = \begin{vmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{vmatrix} = 3\begin{vmatrix} -4 & 3 \\ 4 & -2 \end{vmatrix} - (-2)\begin{vmatrix} 1 & 0 \\ 4 & -2 \end{vmatrix} + 5\begin{vmatrix} 1 & 0 \\ -4 & 3 \end{vmatrix} = 3(-4) - (-2)(-2) + 5(3) = -1$$

Example

- Cofactor expansion along the first row

$$\begin{vmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{vmatrix} = 3\begin{vmatrix} -4 & 3 \\ 4 & -2 \end{vmatrix} - 1\begin{vmatrix} -2 & 3 \\ 5 & -2 \end{vmatrix} + 0\begin{vmatrix} -2 & -4 \\ 5 & 4 \end{vmatrix}$$

$$= 3(-4) - (1)(-11) + 0 = -1$$

Example

- Smart choice of row or column

$$A = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 3 & 1 & 2 & 2 \\ 1 & 0 & -2 & 1 \\ 2 & 0 & 0 & 1 \end{bmatrix}$$

- It's easiest to use cofactor expansion along the second column

$$\det(A) = 1 \cdot \begin{vmatrix} 1 & 0 & -1 \\ 1 & -2 & 1 \\ 2 & 0 & 1 \end{vmatrix} = 1 \cdot (-2) \cdot \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} = -2(1+2) = -6$$

Determinant of an Upper Triangular Matrix

- For simplicity of notation, we prove the result for a 4×4 lower triangular matrix

$$A = \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

$$\det(A) = \begin{vmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & 0 & 0 \\ a_{32} & a_{33} & 0 \\ a_{42} & a_{43} & a_{44} \end{vmatrix} = a_{11}a_{22} \begin{vmatrix} a_{33} & 0 \\ a_{43} & a_{44} \end{vmatrix}$$

$$= a_{11}a_{22}a_{33} \begin{vmatrix} a_{44} \end{vmatrix} = a_{11}a_{22}a_{33}a_{44}$$

Theorem 2.1.2

- If A is an $n \times n$ triangular matrix, then det(A) is the product of the entries on the main diagonal of the matrix: $\det(A) = a_{11}a_{22}\cdots a_{nn}$

Useful Technique for 2x2 and 3x3 Matrices

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad \det = a_{11}a_{22} - a_{12}a_{21}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}$$

2.2

Evaluating Determinants by Row Reduction

Theorem 2.2.1

- Let A be a square matrix. If A has a row of zeros or a column of zeros, then $\det(A) = 0$.
- Proof:
  - Since the determinant of A can be found by a cofactor expansion along any row or column, we can use the row or column of zeros.

$$\det(A) = 0C_1 + 0C_2 + \cdots + 0C_n = 0$$

Theorem 2.2.2

- Let A be a square matrix. Then $\det(A) = \det(A^T)$
- Proof:
  - Since transposing a matrix changes it columns to rows and its rows to columns, the cofactor expansion of A along any row is the same as the cofactor expansion of A^T along the corresponding column. Thus, both have the same determinant.

Theorem 2.2.3 (Elementary Row Operations)

- Let A be an $n \times n$ matrix
  - If B is the matrix that results when a single row or single column of A is multiplied by a scalar k, than $\det(B) = k \det(A)$
  - If B is the matrix that results when two rows or two columns of A are interchanged, then $\det(B) = -\det(A)$
  - If B is the matrix that results when a multiple of one row of A is added to another row or when a multiple column is added to another column, then $\det(B) = \det(A)$

Example

$$\begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = ka_{11}C_{11} + ka_{12}C_{12} + ka_{13}C_{13}$$

$$= k(a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}) = k \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \stackrel{?}{=} - \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\begin{vmatrix} a_{11} + ka_{21} & a_{12} + ka_{22} & a_{13} + ka_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \stackrel{?}{=} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Theorems

- Theorem 2.2.4 (Elementary Matrices)
  - Let E be an $n \times n$ elementary matrix (基本矩陣)
    - If E results from multiplying a row of I_n by k, then $\det(E) = k$
    - If E results from interchanging two rows of I_n, then $\det(E) = -1$
    - If E results from adding a multiple of one row of I_n to another, then $\det(E) = 1$

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 3 \qquad \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix} = -1 \qquad \begin{vmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 1$$

Theorems

- Theorem 2.2.5 (Matrices with Proportional Rows or Columns)
  - If A is a square matrix with two proportional rows or two proportional column, then $\det(A) = 0$

-2 times Row 1 was added to Row 2

$$\begin{vmatrix} 1 & 3 & -2 & 4 \\ 2 & 6 & -4 & 8 \\ 3 & 9 & 1 & 5 \\ 1 & 1 & 4 & 8 \end{vmatrix} = \begin{vmatrix} 1 & 3 & -2 & 4 \\ 0 & 0 & 0 & 0 \\ 3 & 9 & 1 & 5 \\ 1 & 1 & 4 & 8 \end{vmatrix} = 0$$

$$\begin{bmatrix} -1 & 4 \\ -2 & 8 \end{bmatrix} \quad \begin{bmatrix} 1 & -2 & 7 \\ -4 & 8 & 5 \\ 2 & -4 & 3 \end{bmatrix} \quad \begin{bmatrix} 3 & -1 & 4 & -5 \\ 6 & -2 & 5 & 2 \\ 5 & 8 & 1 & 4 \\ -9 & 3 & -12 & 15 \end{bmatrix}$$

Example (Using Row Reduction to Evaluate a Determinant)

- Evaluate det(A) where

$$A = \begin{bmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{bmatrix}$$

- Solution:

$$\det(A) = \begin{vmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{vmatrix} = -\begin{vmatrix} 3 & -6 & 9 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix}$$

The first and second rows of A are interchanged.

$$= -3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix}$$

A common factor of 3 from the first row was taken through the determinant sign

.

Example

$$\det(A) = -3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix}$$

$$= -3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 10 & -5 \end{vmatrix}$$

← -2 times the first row was added to the third row.

$$= -3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & -55 \end{vmatrix}$$

← -10 times the second row was added to the third row

$$= (-3)(-55)\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{vmatrix}$$

← A common factor of -55 from the last row was taken through the determinant sign.

$$= (-3)(-55)(1) = 165$$

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Example

$$A = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 2 & 7 & 0 & 6 \\ 0 & 6 & 3 & 0 \\ 7 & 3 & 1 & -5 \end{bmatrix}$$

- Using column operations to evaluate a determinant
- Put A in lower triangular form by adding -3 times the first column to the fourth to obtain

$$\det(A) = \det \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 7 & 0 & 0 \\ 0 & 6 & 3 & 0 \\ 7 & 3 & 1 & -26 \end{bmatrix} = (1)(7)(3)(-26) = -546$$

Example

$$A = \begin{bmatrix} 3 & 5 & -2 & 6 \\ 1 & 2 & -1 & 1 \\ 2 & 4 & 1 & 5 \\ 3 & 7 & 5 & 3 \end{bmatrix}$$

- By adding suitable multiples of the second row to the remaining rows, we obtain

Cofactor expansion along the first column

$$\det(A) = \begin{vmatrix} 0 & -1 & 1 & 3 \\ 1 & 2 & -1 & 1 \\ 0 & 0 & 3 & 3 \\ 0 & 1 & 8 & 0 \end{vmatrix} = - \begin{vmatrix} -1 & 1 & 3 \\ 0 & 3 & 3 \\ 1 & 8 & 0 \end{vmatrix}$$

$$= - \begin{vmatrix} -1 & 1 & 3 \\ 0 & 3 & 3 \\ 0 & 9 & 3 \end{vmatrix} = -(-1) \begin{vmatrix} 3 & 3 \\ 9 & 3 \end{vmatrix} = -18$$

Add the first row to the third row

Cofactor expansion along the first column

2.3

Properties of Determinants; Cramer's Rule

Basic Properties of Determinant

- Since a common factor of any row of a matrix can be moved through the det sign, and since each of the n row in kA has a common factor of k, we obtain

$$\det(kA) = k^n \det(A)$$

- There is no simple relationship exists between $\det(A)$, $\det(B)$, and $\det(A+B)$ in general.
- In particular, we emphasize that $\det(A+B)$ is usually *not* equal to $\det(A) + \det(B)$.

Example

$$\begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ ka_{21} & ka_{22} & ka_{23} \\ ka_{31} & ka_{32} & ka_{33} \end{vmatrix} = k^3 \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

- Consider

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \quad A + B = \begin{bmatrix} 4 & 3 \\ 3 & 8 \end{bmatrix}$$

- We have det(A) = 1, det(B) = 8, and det(A+B)=23; thus

$$\det(A + B) \neq \det(A) + \det(B)$$

Example

- Consider two matrices that differ only in the second row

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad B = \begin{bmatrix} a_{11} & a_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

$$\det(A) + \det(B) = (a_{11}a_{22} - a_{12}a_{21}) + (a_{11}b_{22} - a_{12}b_{21})$$
$$= a_{11}(a_{22} + b_{22}) - a_{12}(a_{21} + b_{21})$$
$$= \det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{bmatrix}$$

$$\det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \det \begin{bmatrix} a_{11} & a_{12} \\ b_{21} & b_{22} \end{bmatrix} = \det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{bmatrix}$$

Theorems 2.3.1

- Let A, B, and C be $n \times n$ matrices that differ only in a single row, say the r-th, and assume that the r-th row of C can be obtained by adding corresponding entries in the r-th rows of A and B. Then

  $$\det(C) = \det(A) + \det(B)$$

  The same result holds for columns.

- Example

$$\det\begin{bmatrix} 1 & 7 & 5 \\ 2 & 0 & 3 \\ 1+0 & 4+1 & 7+(-1) \end{bmatrix} = \det\begin{bmatrix} 1 & 7 & 5 \\ 2 & 0 & 3 \\ 1 & 4 & 7 \end{bmatrix} + \det\begin{bmatrix} 1 & 7 & 5 \\ 2 & 0 & 3 \\ 0 & 1 & -1 \end{bmatrix}$$

Theorems

- Lemma 2.3.2
  - If B is an $n \times n$ matrix and E is an $n \times n$ elementary matrix, then

$$\det(EB) = \det(E)\det(B)$$

- Remark:
  - If B is an $n \times n$ matrix and $E_1, E_2, \ldots, E_r$, are $n \times n$ elementary matrices, then

$$\det(E_1 E_2 \cdots E_r B) = \det(E_1)\det(E_2) \cdots \det(E_r)\det(B)$$

Proof of Lemma 2.3.2

If B is an $n \times n$ matrix and E is an $n \times n$ elementary matrix, then

$$\det(EB) = \det(E)\det(B)$$

- We shall consider three cases, each depending on the row operation that produces matrix E.
- Case 1. If E results from multiplying a row of I_n by k, then by Theorem 1.5.1, EB results from B by multiplying a row by k; so from Theorem 2.2.3a we have

$$\det(EB) = k\det(B)$$

  From Theorem 2.2.4a, we have $\det(E) = k$, so

$$\det(EB) = \det(E)\det(B)$$

- Cases 2 and 3. E results from interchanging two rows of I_n or from adding a multiple of one row to another.

Theorems

- Theorem 2.3.3 (Determinant Test for Invertibility)
  - A square matrix A is invertible if and only if $\det(A) \neq 0$
- Proof: Let R be the reduced row-echelon form of A.

$$R = E_r \cdots E_2 E_1 A$$

$$\det(R) = \det(E_r) \cdots \det(E_2) \det(E_1) \det(A)$$

From Theorem 2.2.4, the determinants of the elementary matrices are all nonzero. Thus, $\det(A)$ and $\det(R)$ are both zero or both nonzero.

Proof of Theorem 2.3.3

- If A is invertible, then by Theorem 1.6.4, we have $R = I$, so $\det(R) = 1 \neq 0$ and consequently $\det(A) \neq 0$.
- Conversely, if $\det(A) \neq 0$, then $\det(R) \neq 0$, so R cannot have a row of zeros. It follows from Theorem 1.4.3 that $R=I$, so A is invertible by Theorem 1.6.4.

Example: Determinant Test for Invertibility

- Since the first and third rows are proportional, det(A) = 0

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 2 & 4 & 6 \end{bmatrix}$$

- A is not invertible.

Theorems

- Theorem 2.3.4
  - If A and B are square matrices of the same size, then

$$\det(AB) = \det(A)\det(B)$$

- Theorem 2.3.5
  - If A is invertible, then

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

Proof of Theorem 2.3.4

- If the matrix A is not invertible, then by Theorem 1.6.5 neither is the product AB.
- Thus, from Theorem 2.3.3, we have $\det(AB) = 0$ and $\det(A) = 0$, so it follows that $\det(AB) = \det(A)\det(B)$.
- Now assume that A is invertible. By Theorem 1.6.4, the matrix A is expressible as a product of elementary matrices, say

$$A = E_1 E_2 \cdots E_r$$

$$AB = E_1 E_2 \cdots E_r B$$

Proof of Theorem 2.3.4

$$AB = E_1 E_2 \cdots E_r B$$

⇓

$$\det(AB) = \det(E_1)\det(E_2)\cdots\det(E_r)\det(B)$$

⇓

$$\det(AB) = \det(E_1 E_2 \cdots E_r)\det(B)$$

⇓

$$\det(AB) = \det(A)\det(B)$$

Proof of Theorem 2.3.5

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

- Since $A^{-1}A = I$, it follows that $\det(A^{-1}A)=\det(I)$.
- Therefore, we must have $\det(A^{-1})\det(A) = 1$.
- Since $\det(A) \neq 0$, the proof can be completed by dividing through by $\det(A)$.

Example

- If one multiplies the entries in any row by the corresponding cofactors from a ***different*** row, the sum of these products is always zero.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

- Consider the quantity $a_{11}C_{31} + a_{12}C_{32} + a_{13}C_{33} = ?$
- Construct a new matrix *A*' by replacing the third row of *A* with another copy of the first row

$$A' = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \end{bmatrix}$$

Example

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad A' = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \end{bmatrix}$$

- Since the first two rows of A and A' are the same, and since the computations of C_{31}, C_{32}, C_{33}, C_{31}', C_{32}', and C_{33}' involve only entries from the first two rows of A and A', it follows that

$$C_{31} = C'_{31} \qquad C_{32} = C'_{32} \qquad C_{33} = C'_{33}$$

- Since A' has two identical rows, det(A') = 0
- By evaluating det(A') by cofactor expansion along the third row gives

$$\det(A') = a_{11}C'_{31} + a_{12}C'_{32} + a_{13}C'_{33} = a_{11}C_{31} + a_{12}C_{32} + a_{13}C_{33} = 0$$

Definition

- If A is any $n \times n$ matrix, and C_{ij} is the cofactor of a_{ij}, then the matrix is called the ***matrix of cofactors from A*** (餘因子矩陣).

$$\begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}$$

- The transpose of this matrix is called the ***adjoint of A*** (伴隨矩陣) and is denoted by adj(A)

$$\begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \vdots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

Adjoint of a 3x3 Matrix

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{bmatrix}$$

Cofactors of A are

$$\begin{array}{lll} C_{11} = 12 & C_{12} = 6 & C_{13} = -16 \\ C_{21} = 4 & C_{22} = 2 & C_{23} = 16 \\ C_{31} = 12 & C_{32} = -10 & C_{33} = 16 \end{array}$$

The matrix of cofactors is

$$\begin{bmatrix} 12 & 6 & -16 \\ 4 & 2 & 16 \\ 12 & -10 & 16 \end{bmatrix}$$

The adjoint of A

$$\begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix}$$

Theorems

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

$$A\,\mathrm{adj}(A) = \det(A)\,I$$

- Theorem 2.3.6 (Inverse of a Matrix using its Adjoint)
  - If A is an invertible matrix, then $A^{-1} = \dfrac{1}{\det(A)}\mathrm{adj}(A)$

Proof of Theorem 2.3.6

If A is an invertible matrix, then $A^{-1} = \frac{1}{\det(A)} \text{adj}(A)$

- We show first that $A\text{adj}(A) = \det(A)I$

$$Aadj(A) = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{j1} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{j2} & \cdots & C_{n2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{jn} & \cdots & C_{nn} \end{bmatrix}$$

- The entry in the ith row and jth column of $A\text{adj}(A)$ is

$$a_{i1}C_{j1} + a_{i2}C_{j2} + \cdots + a_{in}C_{jn}$$

Proof of Theorem 2.3.6

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$$a_{i1}C_{j1} + a_{i2}C_{j2} + \cdots + a_{in}C_{jn}$$

- If $i=j$, then it is the cofactor expansion of det(A) along the ith row of A.
- If $i \neq j$, then the a's and the cofactors come from different rows of A, so the value is zero. Therefore,

$$Aadj(A) = \begin{bmatrix} \det(A) & 0 & \cdots & 0 \\ 0 & \det(A) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \det(A) \end{bmatrix} = \det(A)I$$

- Since A is invertible, det(A)$\neq$ 0. Therefore

$$\frac{1}{\det(A)}[Aadj(A)] = I \Rightarrow A\left[\frac{1}{\det(A)}adj(A)\right] = I$$

- Multiplying both sides on the left by A^{-1} yields $$A^{-1} = \frac{1}{\det(A)}adj(A)$$

Example

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{bmatrix}$$

The adjoint of $A = \begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix}$

$$A^{-1} = \frac{1}{\det(A)} adj(A) = \frac{1}{64} \begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix}$$

Theorem 2.3.7 (Cramer's Rule)

- If $A\mathbf{x} = \mathbf{b}$ is a system of n linear equations in n unknowns such that $\det(A) \neq 0$, then the system has a unique solution. This solution is

$$x_1 = \frac{\det(A_1)}{\det(A)}, \quad x_2 = \frac{\det(A_2)}{\det(A)}, \cdots, x_n = \frac{\det(A_n)}{\det(A)}$$

where A_j is the matrix obtained by replacing the entries in the jth column of A by the entries in the matrix $\mathbf{b} = [b_1 \; b_2 \; \cdots \; b_n]^T$

Proof of Theorem 2.3.7

- If $\det(A) \neq 0$, then *A* is invertible, and by Theorem 1.6.2, $\boldsymbol{x} = A^{-1}\boldsymbol{b}$ is the unique solution of $A\boldsymbol{x} = \boldsymbol{b}$. Therefore, by Theorem 2.3.6, we have

$$\boldsymbol{x} = A^{-1}\boldsymbol{b} = \frac{1}{\det(A)}\operatorname{adj}(A)\boldsymbol{b}$$

$$= \frac{1}{\det(A)}\begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Proof of Theorem 2.3.7

$$\boldsymbol{x} = \frac{1}{\det(A)} \begin{bmatrix} b_1C_{11} + b_2C_{21} + \cdots + b_nC_{n1} \\ b_1C_{12} + b_2C_{22} + \cdots + b_nC_{n2} \\ \vdots \\ b_1C_{1n} + b_2C_{2n} + \cdots + b_nC_{nn} \end{bmatrix}$$

- The entry in the *j*th row of ***x*** is therefore

$$x_j = \frac{b_1C_{1j} + b_2C_{2j} + \cdots + b_nC_{nj}}{\det(A)}$$

- Now let

$$A_j = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j-1} & b_1 & a_{1j+1} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j-1} & b_2 & a_{2j+1} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \\ a_{n1} & a_{n2} & \cdots & a_{nj-1} & b_n & a_{nj+1} & \cdots & a_{nn} \end{bmatrix}$$

Proof of Theorem 2.3.7

- Since A_j differs form A only in the jth column, it follows that the cofactors of entries b_1, b_2, …, b_n in A_j are the same as the cofactors of the corresponding entries in the jth column of A.
- The cofactor expansion of det(A_j) along the jth column is therefore $\det(A_j) = b_1C_{1j} + b_2C_{2j} + \cdots + b_nC_{nj}$
- Substituting this result gives

$$x_j = \frac{\det(A_j)}{\det(A)}$$

Example

- Use Cramer's rule to solve

$$\begin{aligned} x_1 + \qquad + 2x_3 &= 6 \\ -3x_1 + 4x_2 + 6x_3 &= 30 \\ -x_1 - 2x_2 + 3x_3 &= 8 \end{aligned}$$

- Since

$$A = \begin{bmatrix} 1 & 0 & 2 \\ -3 & 4 & 6 \\ -1 & -2 & 3 \end{bmatrix}, A_1 = \begin{bmatrix} 6 & 0 & 2 \\ 30 & 4 & 6 \\ 8 & -2 & 3 \end{bmatrix}, A_2 = \begin{bmatrix} 1 & 6 & 2 \\ -3 & 30 & 6 \\ -1 & 8 & 3 \end{bmatrix}, A_3 = \begin{bmatrix} 1 & 0 & 6 \\ -3 & 4 & 30 \\ -1 & -2 & 8 \end{bmatrix}$$

- Thus,

$$x_1 = \frac{\det(A_1)}{\det(A)} = \frac{-40}{44} = \frac{-10}{11}, x_2 = \frac{\det(A_2)}{\det(A)} = \frac{72}{44} = \frac{18}{11}, x_3 = \frac{\det(A_3)}{\det(A)} = \frac{152}{44} = \frac{38}{11}$$

Theorem 2.3.8 (Equivalent Statements)

- If A is an $n \times n$ matrix, then the following are equivalent
  - A is invertible.
  - $A\mathbf{x} = \mathbf{0}$ has only the trivial solution
  - The reduced row-echelon form of A as I_n
  - A is expressible as a product of elementary matrices
  - $A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix $\mathbf{b}$
  - $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $n \times 1$ matrix $\mathbf{b}$
  - $\det(A) \neq 0$