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# Chapter 1

## Systems of Linear Equations and Matrices

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# Outline

- 1.1 Introduction to Systems of Linear Equations
- 1.2 Gaussian Elimination
- 1.3 Matrices and Matrix Operations
- 1.4 Inverse; Algebraic Properties of Matrices
- 1.5 Elementary Matrices and a Method for Finding  $A^{-1}$
- 1.6 More on Linear Systems and Invertible Matrices
- 1.7 Diagonal, Triangular, and Symmetric Matrices

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# 1.1

## Introduction to Systems of Linear Equations

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# Linear Equations

- Any straight line in  $xy$ -plane can be represented algebraically by an equation of the form:

$$a_1x + a_2y = b$$

- General form: Define a **linear equation** in the  $n$  variables  $x_1, x_2, \dots, x_n$ :

$$a_1x_1 + a_2x_2 + \dots + a_nx_n = b$$

where  $a_1, a_2, \dots, a_n$  and  $b$  are real constants.

- The variables in a linear equation are sometimes called **unknowns**.

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# Example (Linear Equations)

- The equations  $x + 3y = 7$ ,  $y = \frac{1}{2}x + 3z + 1$ , and  $x_1 - 2x_2 - 3x_3 + x_4 = 7$  are linear
  - A linear equation does not involve any products or roots of variables
  - All variables occur only to the first power and do not appear as arguments for trigonometric, logarithmic, or exponential functions.
- The equations  $x + 3\sqrt{y} = 5$ ,  $3x + 2y - z + xz = 4$ , and  $y = \sin x$  are *not* linear
- A **solution** of a linear equation is a sequence of  $n$  numbers  $s_1, s_2, \dots, s_n$  such that the equation is satisfied.
- The set of all solutions of the equation is called its **solution set** or **general solution** (通解) of the equation.

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# Example

- Find the solution of  $x_1 - 4x_2 + 7x_3 = 5$
- Solution:
  - We can assign arbitrary values to any two variables and solve for the third variable
  - For example

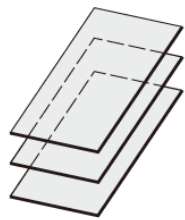
$$x_1 = 5 + 4s - 7t, \quad x_2 = s, \quad x_3 = t$$

where  $s, t$  are arbitrary values

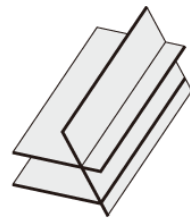


# Linear Systems

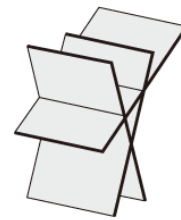
- Example of a linear system of three equations in three unknowns



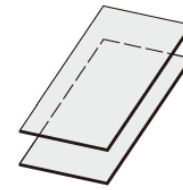
無解  
(三個平行平面；  
沒有共同交集)



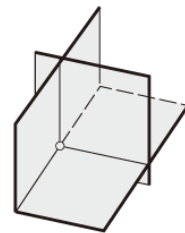
無解  
(兩個平行平面；  
沒有共同交集)



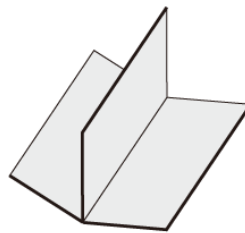
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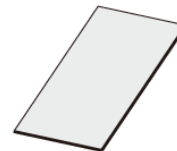
無解  
(二重合平面平行第三個  
平面；沒有共同交集)



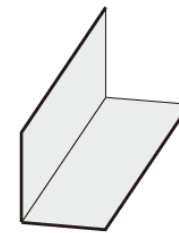
一組解  
(交集是一點)



無限多組解  
(交集是一直線)



無限多組解  
(三個平面重合；  
交集是一個平面)



無限多組解  
(二重合平面；  
交集是一直線)

▲ 圖 1.1.2



# Example

$$\begin{aligned}x - y + 2z &= 5 \\2x - 2y + 4z &= 10 \\3x - 3y + 6z &= 15\end{aligned}$$

After elimination  $\Rightarrow x - y + 2z = 5$

General solution:

$$\begin{aligned}x &= 5 + r - 2s \\y &= r \\z &= s\end{aligned}$$

- The three planes coincide!





# Example (Using Elementary Row Operations)

$$\begin{array}{rcl}
 x + y + 2z = 9 & & x + y + 2z = 9 \\
 2x + 4y - 3z = 1 & \longrightarrow & 2y - 7z = -17 \\
 3x + 6y - 5z = 0 & & 3x + 6y - 5z = 0
 \end{array}
 \longrightarrow
 \begin{array}{rcl}
 x + y + 2z = 9 & & x + y + 2z = 9 \\
 2y - 7z = -17 & \longrightarrow & 2y - 7z = -17 \\
 3y - 11z = -27 & & 3y - 11z = -27
 \end{array}
 \longrightarrow
 \begin{array}{rcl}
 x + y + 2z = 9 & & x + y + 2z = 9 \\
 y - \frac{7}{2}z = -\frac{17}{2} & & y - \frac{7}{2}z = -\frac{17}{2} \\
 3y - 11z = 0 & & 3y - 11z = 0
 \end{array}$$

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{bmatrix}
 \longrightarrow
 \begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 3 & 6 & -5 & 0 \end{bmatrix}
 \longrightarrow
 \begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 3 & -11 & -27 \end{bmatrix}
 \longrightarrow
 \begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 3 & -11 & -27 \end{bmatrix}$$

$$\begin{array}{rcl}
 x + y + 2z = 9 & & x + y + 2z = 9 \\
 y - \frac{7}{2}z = -\frac{17}{2} & \longrightarrow & y - \frac{7}{2}z = -\frac{17}{2} \\
 -\frac{1}{2}z = -\frac{3}{2} & & z = 3
 \end{array}
 \longrightarrow
 \begin{array}{rcl}
 x + y + 2z = 9 & & x + y + 2z = 9 \\
 y - \frac{7}{2}z = -\frac{17}{2} & \longrightarrow & y - \frac{7}{2}z = -\frac{17}{2} \\
 z = 3 & & z = 3
 \end{array}
 \longrightarrow
 \begin{array}{rcl}
 x + \frac{11}{2}z = \frac{35}{2} & & x + \frac{11}{2}z = \frac{35}{2} \\
 y - \frac{7}{2}z = -\frac{17}{2} & \longrightarrow & y - \frac{7}{2}z = -\frac{17}{2} \\
 z = 3 & & z = 3
 \end{array}
 \longrightarrow
 \begin{array}{rcl}
 x = 1 & & x = 1 \\
 y = 2 & & y = 2 \\
 z = 3 & & z = 3
 \end{array}$$

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & -\frac{1}{2} & -\frac{3}{2} \end{bmatrix}
 \longrightarrow
 \begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & 1 & 3 \end{bmatrix}
 \longrightarrow
 \begin{bmatrix} 1 & 0 & \frac{11}{2} & \frac{35}{2} \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & 1 & 3 \end{bmatrix}
 \longrightarrow
 \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

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1.2

Gaussian Elimination

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# Echelon Forms

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

- A matrix which has the following properties is in **reduced row-echelon form** (as in the previous example) (簡約列-梯型)
  - If a row does not consist entirely of zeros, then the first nonzero number in the row is a 1. We call this a **leading 1**. (首項1)
  - If there are any rows that consist entirely of zeros, then they are grouped together at the bottom of the matrix.
  - In any two successive rows that do not consist entirely of zeros, the leading 1 in the lower row occurs farther to the right than the leading 1 in the higher row.
  - Each *column* that contains a leading 1 has zeros everywhere else.
- A matrix that has the *first three properties* is said to be in **row-echelon form**. (列-梯型)
- Note: A matrix in reduced row-echelon form is of necessity in row-echelon form, but not conversely.

# Example

- Reduced row-echelon form:

$$\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

- Row-echelon form:

$$\begin{bmatrix} 1 & 4 & -3 & 7 \\ 0 & 1 & 6 & 2 \\ 0 & 0 & 1 & 5 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 2 & 6 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

# Example

- All matrices of the following types are in **row-echelon form** (any real numbers substituted for the \*'s. ) :

$$\begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 1 & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

- All matrices of the following types are in **reduced row-echelon form** (any real numbers substituted for the \*'s. ) :

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & * & 0 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 1 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$



# Example

- Suppose that the augmented matrix for a linear system in the unknowns  $x, y, z$  has been reduced as

$$\begin{bmatrix} 1 & -5 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{array}{l} \Rightarrow \quad x - 5y + z = 4 \\ \quad \quad 0x + 0y + 0z = 0 \end{array} \quad \Rightarrow \quad x = 4 + 5y - z$$

General solution:

$$\begin{array}{l} x = 4 + 5s - t \\ y = s \\ z = t \end{array}$$

$s$  and  $t$  can be arbitrary values

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# Elimination Methods

- A step-by-step **elimination** procedure that can be used to reduce any matrix to reduced row-echelon form

$$\begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

# Elimination Methods

$$\begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

- Step1. Locate the leftmost column that does not consist entirely of zeros.

$$\begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

**Leftmost nonzero column**

- Step2. Interchange the top row with another row, to bring a nonzero entry to top of the column found in Step1

$$\begin{bmatrix} 2 & 4 & -10 & 6 & 12 & 28 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

**The 1th and 2th rows in the preceding matrix were interchanged.**

# Elimination Methods

- Step3. If the entry that is now at the top of the column found in Step1 is  $a$ , multiply the first row by  $1/a$  in order to introduce a leading 1.

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

← **The 1st row of the preceding matrix was multiplied by  $1/2$ .**

- Step4. Add suitable multiples of the top row to the rows below so that all entries below the leading 1 become zeros

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{bmatrix}$$

← **-2 times the 1st row of the preceding matrix was added to the 3rd row.**

# Elimination Methods

- Step5. Now cover the top row in the matrix and begin again with Step1 applied to the submatrix that remains. Continue in this way until the entire matrix is in row-echelon form

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{bmatrix}$$

**Leftmost nonzero  
column in the submatrix**

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & -6 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{bmatrix}$$

**The 1st row in the submatrix  
was multiplied by  $-\frac{1}{2}$  to  
introduce a leading 1.**

# Elimination Methods

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & -6 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 1 \end{bmatrix}$$

**-5 times the 1st row of the submatrix was added to the 2nd row of the submatrix to introduce a zero below the leading 1.**

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & -6 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 1 \end{bmatrix}$$

**The top row in the submatrix was covered, and we returned again Step1.**

**Leftmost nonzero column in the new submatrix**

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & -6 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

**The first (and only) row in the new submatrix was multiplied by 2 to introduce a leading 1.**

# Elimination Methods

- Step 6. Beginning with the last nonzero row and working upward, add suitable multiples of each row to the rows above to introduce zeros above the leading 1's.

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

← **7/2 times the third row was added to the second row**

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

← **-6 times the third row was added to the first row**

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 0 & 7 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

← **5 times the second row was added to the first row**

The **last** matrix is in **reduced row-echelon form**

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# Elimination Methods

- Step1~Step5: the above procedure produces a row-echelon form and is called **Gaussian elimination**
- Step1~Step6: the above procedure produces a reduced row-echelon form and is called **Gaussian-Jordan elimination**
- Every matrix has a **unique reduced row-echelon** form but a row-echelon form of a given matrix is not unique
- Back-Substitution
  - It is sometimes preferable to solve a system of linear equations by using Gaussian elimination to bring the augmented matrix into row-echelon form **without continuing all the way to the reduced row-echelon form.**
  - When this is done, the corresponding system of equations can be solved by a technique called **back-substitution**

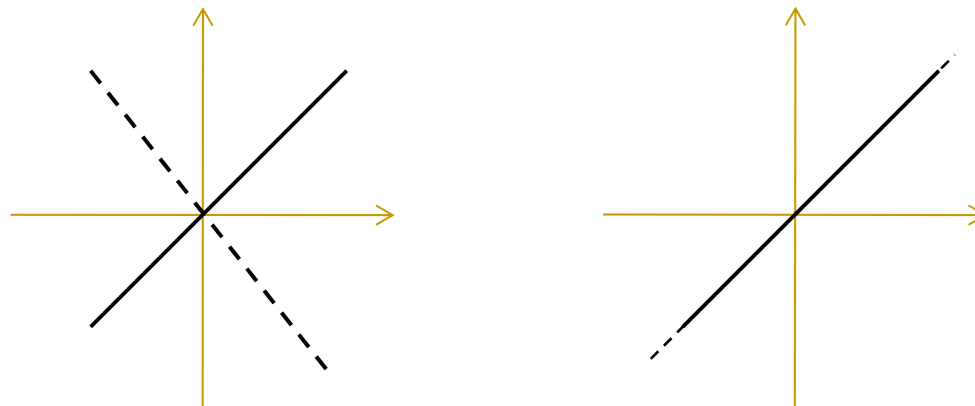




# Example

- A homogeneous linear system of two equations in two unknowns

$$\begin{aligned}a_1x + b_1y &= 0 \\ a_2x + b_2y &= 0\end{aligned}$$



# Example (Gauss-Jordan Elimination)

- Solve the homogeneous system of linear equations by Gauss-Jordan elimination

$$2x_1 + 2x_2 - x_3 + x_5 = 0$$

$$-x_1 - x_2 + 2x_3 - 3x_4 + x_5 = 0$$

$$x_1 + x_2 - 2x_3 - x_5 = 0$$

$$x_3 + x_4 + x_5 = 0$$

- The augmented matrix

$$\begin{bmatrix} 2 & 2 & -1 & 0 & 1 & 0 \\ -1 & -1 & 2 & -3 & 1 & 0 \\ 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

- Reducing this matrix to reduced row-echelon form

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- The general solution is

$$x_1 = -s - t, x_2 = s$$

$$x_3 = -t, x_4 = 0, x_5 = t$$

- Note: the trivial solution is obtained when  $s = t = 0$

# Example (Gauss-Jordan Elimination)

- Two important points:
  - None of the three row operations alters the final column of zeros, so the system of equations corresponding to the reduced row-echelon form of the augmented matrix must also be a homogeneous system.
  - If the given homogeneous system has  $m$  equations in  $n$  unknowns with  $m < n$ , and there are  $r$  nonzero rows in reduced row-echelon form of the augmented matrix, we will have  $r < n$ . It will have the form:

$$\begin{array}{rcl} \cdots x_{k1} & + \sum () = 0 & x_{k1} = -\sum () \\ \cdots x_{k2} & + \sum () = 0 & x_{k2} = -\sum () \\ \ddots & \vdots & \vdots \\ x_{kr} & + \sum () = 0 & x_{kr} = -\sum () \end{array}$$

$$\begin{array}{rclcl} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n & = & b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n & = & b_2 \\ \vdots & & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n & = & b_m \end{array}$$

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# Theorem

- Theorem 1.2.1
  - If a homogeneous system has  $n$  unknowns, and if the reduced row echelon form of its augmented matrix has  $r$  nonzero rows, then the system has  $n-r$  free variables.
- Theorem 1.2.2
  - A homogeneous system of linear equations with more unknowns than equations has **infinitely many solutions**.

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# Remarks

- Every matrix has a unique reduced row echelon form
- Row echelon forms are not unique
- Although row echelon forms are not unique, all row echelon forms of a matrix  $A$  have the same number of zero rows, and the leading 1's always occur in the same positions in the row echelon forms of  $A$ .

$$A = \begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix} \quad \longrightarrow \quad \begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & -6 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

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# 1.3

## Matrices and Matrix Operations

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# Definition and Notation

- A **matrix** is a rectangular array of numbers. The numbers in the array are called the **entries** (元素) in the matrix
- A general  $m \times n$  matrix  $A$  is denoted as

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

- The entry that occurs in row  $i$  and column  $j$  of matrix  $A$  will be denoted  $a_{ij}$  or  $\langle A \rangle_{ij}$ . If  $a_{ij}$  is real number, it is common to be referred as **scalars** (純量)
- The preceding matrix can be written as  $[a_{ij}]_{m \times n}$  or  $[a_{ij}]$
- A matrix  $A$  with  $n$  rows and  $n$  columns is called a **square matrix of order  $n$**



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# Examples of Matrices

$$\begin{bmatrix} 1 & 2 \\ 3 & 0 \\ -1 & 4 \end{bmatrix} \quad [2 \ 1 \ 0 \ -3] \quad \begin{bmatrix} \pi & -\sqrt{2} \\ \frac{3}{2} & 1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad [3]$$

- A matrix with only one row is called a *row matrix* (or a *row vector*).
- A matrix with only one column is called a *column matrix* (or a *column vector*).

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# Sum, Difference, and Product

- Two matrices are defined to be **equal** if they have the same size and their corresponding entries are equal
  - If  $A = [a_{ij}]$  and  $B = [b_{ij}]$  have the same size, then  $A = B$  if and only if  $a_{ij} = b_{ij}$  for all  $i$  and  $j$
- If  $A$  and  $B$  are matrices of the same size, then the **sum**  $A + B$  is the matrix obtained by adding the entries of  $B$  to the corresponding entries of  $A$ .
- The **difference**  $A - B$  is the matrix obtained by subtracting the entries of  $B$  from the corresponding entries of  $A$
- If  $A$  is any matrix and  $c$  is any scalar, then the **product**  $cA$  is the matrix obtained by multiplying each entry of the matrix  $A$  by  $c$ . The matrix  $cA$  is said to be the **scalar multiple** of  $A$ 
  - If  $A = [a_{ij}]$ , then  $\langle cA \rangle_{ij} = c \langle A \rangle_{ij} = ca_{ij}$

# Example

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 3 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 2 & 7 \\ -1 & 3 & -5 \end{bmatrix}$$

$$A + B = \begin{bmatrix} 2 & 5 & 11 \\ 0 & 6 & -4 \end{bmatrix} \quad 2A = \begin{bmatrix} 4 & 6 & 8 \\ 2 & 6 & 2 \end{bmatrix}$$

$$A - B = \begin{bmatrix} 2 & 1 & -3 \\ 2 & 0 & 6 \end{bmatrix}$$

linear combination:

$$2A + 3B = \begin{bmatrix} 4 & 6 & 8 \\ 2 & 6 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 6 & 21 \\ -3 & 9 & -15 \end{bmatrix} = \begin{bmatrix} 4 & 12 & 29 \\ -1 & 15 & -13 \end{bmatrix}$$

# Product of Matrices

- If  $A$  is an  $m \times r$  matrix and  $B$  is an  $r \times n$  matrix, then the **product**  $AB$  is the  $m \times n$  matrix whose entries are determined as follows.
- To find the entry in row  $i$  and column  $j$  of  $AB$ , single out row  $i$  from the matrix  $A$  and column  $j$  from the matrix  $B$ . Multiply the corresponding entries from the row and column together and then add up the resulting products

- That is,  $(AB)_{m \times n} = A_{m \times r} B_{r \times n}$

$$AB = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1r} \\ a_{21} & a_{22} & \cdots & a_{2r} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ir} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mr} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ b_{r1} & b_{r2} & \cdots & b_{rj} & \cdots & b_{rn} \end{bmatrix}$$

the entry  $\langle AB \rangle_{ij}$  in row  $i$  and column  $j$  of  $AB$  is given by

$$\langle AB \rangle_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \dots + a_{ir}b_{rj}$$

# Product of Matrices

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix} = \begin{bmatrix} \blacksquare & \blacksquare & \blacksquare & \blacksquare \\ \blacksquare & \blacksquare & 26 & \blacksquare \end{bmatrix}$$

$$2 \times 4 + 6 \times 3 + 0 \times 5 = 26$$

$$AB = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix} = \begin{bmatrix} \blacksquare & \blacksquare & \blacksquare & 13 \\ \blacksquare & \blacksquare & \blacksquare & \blacksquare \end{bmatrix}$$

$$1 \times 3 + 2 \times 1 + 4 \times 2 = 13$$

---

# Example

- Determining whether a product is defined

$$A_{3 \times 4} \quad B_{4 \times 7} \quad C_{7 \times 3}$$

- $AB$  is defined and is a  $3 \times 7$  matrix;  $BC$  is defined and is a  $4 \times 3$  matrix; and  $CA$  is defined and is a  $7 \times 4$  matrix.
- The products  $AC$ ,  $CB$ , and  $BA$  are all undefined.

# Example

- If  $A = [a_{ij}]$  is a  $m \times r$  matrix and  $B = [b_{ij}]$  is an  $r \times n$  matrix, then the entry  $(AB)_{ij}$  is given by

$$AB = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1r} \\ a_{21} & a_{22} & \cdots & a_{2r} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ir} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mr} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ b_{r1} & b_{r2} & \cdots & b_{rj} & \cdots & b_{rn} \end{bmatrix}$$

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ir}b_{rj}$$

# Partitioned Matrices

- A matrix can be subdivided or **partitioned** into smaller matrices by inserting horizontal and vertical rules between selected rows and columns
- For example, three possible partitions of a  $3 \times 4$  matrix  $A$ :

- The partition of  $A$  into four **submatrices**  $A_{11}$ ,  $A_{12}$ ,  $A_{21}$ , and  $A_{22}$

$$A = \left[ \begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ \hline a_{31} & a_{32} & a_{33} & a_{34} \end{array} \right] = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

- The partition of  $A$  into its row matrices  $\mathbf{r}_1$ ,  $\mathbf{r}_2$ , and  $\mathbf{r}_3$

$$A = \left[ \begin{array}{cccc} a_{11} & a_{12} & a_{13} & a_{14} \\ \hline a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{array} \right] = \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{r}_3 \end{bmatrix}$$

- The partition of  $A$  into its column matrices  $\mathbf{c}_1$ ,  $\mathbf{c}_2$ ,  $\mathbf{c}_3$ , and  $\mathbf{c}_4$

$$A = \left[ \begin{array}{c|c|c|c} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ \hline a_{31} & a_{32} & a_{33} & a_{34} \end{array} \right] = [\mathbf{c}_1 \quad \mathbf{c}_2 \quad \mathbf{c}_3 \quad \mathbf{c}_4]$$



# Multiplication by Columns and by Rows

- It is possible to compute a particular row or column of a matrix product  $AB$  without computing the entire product:

$$j\text{th column matrix of } AB = A[j\text{th column matrix of } B]$$

$$i\text{th row matrix of } AB = [i\text{th row matrix of } A]B$$

- If  $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_m$  denote the row matrices of  $A$  and  $\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n$  denote the column matrices of  $B$ , then

$$AB = A[\mathbf{b}_1 \quad \mathbf{b}_2 \quad \cdots \quad \mathbf{b}_n] = [A\mathbf{b}_1 \quad A\mathbf{b}_2 \quad \cdots \quad A\mathbf{b}_n]$$
$$AB = \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{a}_m \end{bmatrix} B = \begin{bmatrix} \mathbf{a}_1 B \\ \mathbf{a}_2 B \\ \vdots \\ \mathbf{a}_m B \end{bmatrix}$$

# Example

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix} \quad AB = \begin{bmatrix} 12 & 27 & 30 & 13 \\ 8 & -4 & 26 & 12 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 7 \end{bmatrix} = \begin{bmatrix} 27 \\ -4 \end{bmatrix} \leftarrow \text{Second column of } AB$$

$A$       Second column of  $B$        $Ab_2$

$$\begin{bmatrix} 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix} = \begin{bmatrix} 12 & 27 & 30 & 13 \end{bmatrix}$$

First row of  $A$        $B$       First row of  $AB$        $a_1B$

# Matrix Products as Linear Combinations

- Let

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

- Then

$$A\mathbf{x} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix} = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

- The product  $A\mathbf{x}$  of a matrix  $A$  with a column matrix  $\mathbf{x}$  is a **linear combination** of the column matrices of  $A$  with the coefficients coming from the matrix  $\mathbf{x}$

# Example

The matrix product

$$\begin{bmatrix} -1 & 3 & 2 \\ 1 & 2 & -3 \\ 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}$$

can be written as the linear combination of column matrices

$$2 \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} - 1 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}$$

The matrix product

$$\begin{bmatrix} 1 & -9 & -3 \end{bmatrix} \begin{bmatrix} -1 & 3 & 2 \\ 1 & 2 & -3 \\ 2 & 1 & -2 \end{bmatrix} = \begin{bmatrix} -16 & -18 & 35 \end{bmatrix}$$

can be written as the linear combination of row matrices

$$1[-1 \quad 3 \quad 2] - 9[1 \quad 2 \quad -3] - 3[2 \quad 1 \quad -2] = [-16 \quad -18 \quad 35]$$

## Example (Columns of a Product $AB$ as Linear Combinations)

We showed in Example 5 that

$$AB = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix} = \begin{bmatrix} 12 & 27 & 30 & 13 \\ 8 & -4 & 26 & 12 \end{bmatrix}$$

The column matrices of  $AB$  can be expressed as linear combinations of the column matrices of  $A$  as follows:

$$\begin{bmatrix} 12 \\ 8 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 0 \begin{bmatrix} 2 \\ 6 \end{bmatrix} + 2 \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 27 \\ -4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 6 \end{bmatrix} + 7 \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 30 \\ 26 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ 6 \end{bmatrix} + 5 \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 13 \\ 12 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 6 \end{bmatrix} + 2 \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$



# Matrix Form of a Linear System

- Consider any system of  $m$  linear equations in  $n$  unknowns:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned} \quad \Rightarrow \quad \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \quad \Rightarrow \quad \mathbf{Ax} = \mathbf{b}$$

- The matrix  $A$  is called the **coefficient matrix** of the system
- The **augmented matrix** of the system is given by

$$[A \mid \mathbf{b}] = \left[ \begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array} \right]$$

---

# Matrices Defining Functions

- We can view  $A$  as defining a rule that shows how a given  $x$  is mapped into a corresponding  $y$ .

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad x = \begin{bmatrix} a \\ b \end{bmatrix}$$
$$y = Ax = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ -b \end{bmatrix}$$

- The effect of multiplying  $A$  by a column vector is to change the sign of the second entry of the column vector.

---

# Matrices Defining Functions

$$B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad y = Bx = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} b \\ -a \end{bmatrix}$$

- The effect of multiplying  $B$  by a column vector is to interchange the first and second entries of the column vector, also changing the sign of the first entry.



# Transpose (轉置矩陣)

- If  $A$  is any  $m \times n$  matrix, then the **transpose of  $A$** , denoted by  $A^T$ , is defined to be the  $n \times m$  matrix that results from interchanging the rows and columns of  $A$ 
  - That is, the first column of  $A^T$  is the first row of  $A$ , the second column of  $A^T$  is the second row of  $A$ , and so forth
- If  $A$  is a square matrix, then the **trace (跡數) of  $A$** , denoted by  $\text{tr}(A)$ , is defined to be the **sum** of the entries on the main diagonal of  $A$ . The trace of  $A$  is undefined if  $A$  is not a square matrix.
  - For an  $n \times n$  matrix  $A = [a_{ij}]$ ,  $\text{tr}(A) = \sum_{i=1}^n a_{ii}$

---

# Example

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} \quad B = \begin{bmatrix} 2 & 3 \\ 1 & 4 \\ 5 & 6 \end{bmatrix} \quad C = [1 \ 3 \ 5] \quad D = [4]$$

$$A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \\ a_{14} & a_{24} & a_{34} \end{bmatrix} \quad B^T = \begin{bmatrix} 2 & 1 & 5 \\ 3 & 4 & 6 \end{bmatrix} \quad C^T = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} \quad D^T = [4]$$

---

# Example

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$B = \begin{bmatrix} -1 & 2 & 7 & 0 \\ 3 & 5 & -8 & 4 \\ 1 & 2 & 7 & -3 \\ 4 & -2 & 1 & 0 \end{bmatrix}$$

$$\text{tr}(A) = a_{11} + a_{22} + a_{33}$$

$$\text{tr}(B) = -1 + 5 + 7 + 0 = 11$$

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# 1.4

## Inverse; Algebraic Properties of Matrices

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# Properties of Matrix Operations

- For real numbers  $a$  and  $b$ , we always have  $ab = ba$ , which is called the *commutative law for multiplication*. For matrices, however,  $AB$  and  $BA$  need not be equal.
- Equality can fail to hold for three reasons:
  - The product  $AB$  is defined but  $BA$  is undefined.
  - $AB$  and  $BA$  are both defined but have different sizes.
  - It is possible to have  $AB \neq BA$  even if both  $AB$  and  $BA$  are defined and have the same size.

---

# Example

$$A = \begin{bmatrix} -1 & 0 \\ 2 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$$

$$AB = \begin{bmatrix} -1 & -2 \\ 11 & 4 \end{bmatrix} \quad BA = \begin{bmatrix} 3 & 6 \\ -3 & 0 \end{bmatrix}$$

  $AB \neq BA$

---

# Theorem 1.4.1

## (Properties of Matrix Arithmetic)

- Assuming that the sizes of the matrices are such that the indicated operations can be performed, the following rules of matrix arithmetic are valid:

- $A + B = B + A$  (commutative law for addition)

- $A + (B + C) = (A + B) + C$  (associative law for addition)

- $A(BC) = (AB)C$  (associative law for multiplication)

- $A(B + C) = AB + AC$  (left distributive law)

- $(B + C)A = BA + CA$  (right distributive law)

- $A(B - C) = AB - AC,$   $(B - C)A = BA - CA$

- $a(B + C) = aB + aC,$   $a(B - C) = aB - aC$

- $(a+b)C = aC + bC,$   $(a-b)C = aC - bC$

- $a(bC) = (ab)C,$   $a(BC) = (aB)C = B(aC)$

## Proof (d)

$$A(B + C) = AB + AC$$

- We must show that  $A(B+C)$  and  $AB+AC$  have the same size and that corresponding entries are equal.
- To form  $A(B+C)$ , the matrices  $B$  and  $C$  must have the same size, say  $m \times n$ , and the matrix  $A$  must then have  $m$  columns, so its size must be of the form  $r \times m$ . This makes  $A(B+C)$  an  $r \times n$  matrix.
- It follows that  $AB+AC$  is also an  $r \times n$  matrix.



## Proof (d)

$$A(B + C) = AB + AC$$

- Suppose that  $A = [a_{ij}]$ ,  $B = [b_{ij}]$ , and  $C = [c_{ij}]$ . We want to show

$$[A(B + C)]_{ij} = [AB + AC]_{ij}$$

- From the definitions of matrix addition and matrix multiplication, we have

$$\begin{aligned} [A(B + C)]_{ij} &= a_{i1}(b_{1j} + c_{1j}) + a_{i2}(b_{2j} + c_{2j}) + \cdots + a_{im}(b_{mj} + c_{mj}) \\ &= (a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{im}b_{mj}) + (a_{i1}c_{1j} + a_{i2}c_{2j} + \cdots + a_{im}c_{mj}) \\ &= [AB]_{ij} + [AC]_{ij} \\ &= [AB + AC]_{ij} \end{aligned}$$

# Example

As an illustration of the associative law for matrix multiplication, consider

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$$

Then

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ 20 & 13 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad BC = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 10 & 9 \\ 4 & 3 \end{bmatrix}$$

Thus,

$$(AB)C = \begin{bmatrix} 8 & 5 \\ 20 & 13 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 18 & 15 \\ 46 & 39 \\ 4 & 3 \end{bmatrix}$$

and

$$A(BC) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 10 & 9 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 18 & 15 \\ 46 & 39 \\ 4 & 3 \end{bmatrix}$$

so  $(AB)C = A(BC)$ , as guaranteed by Theorem 1.4.1c.

# Zero Matrices (零矩陣)

- A matrix, all of whose entries are zero, is called a **zero matrix**
- A zero matrix will be denoted by  $0$
- If it is important to emphasize the size, we shall write  $0_{m \times n}$  for the  $m \times n$  zero matrix.
- In keeping with our convention of using **boldface symbols** for matrices with one column, we will denote a zero matrix with one column by  $\mathbf{0}$
- Theorem 1.4.2 (Properties of Zero Matrices)
  - Assuming that the sizes of the matrices are such that the indicated operations can be performed, the following rules of matrix arithmetic are valid
    - $A + 0 = 0 + A = A$
    - $A - A = 0$
    - $0 - A = -A$
    - $A0 = 0; 0A = 0$

# Cancellation Law

- For real numbers:
  - If  $ab=ac$  and  $a \neq 0$ , then  $b = c$
  - If  $ab = 0$ , then at least one of the factors on the left is 0.
- It fails in matrix operation

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix} \quad C = \begin{bmatrix} 2 & 5 \\ 3 & 4 \end{bmatrix}$$

$$AB = AC = \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix} \quad \text{but } B \neq C$$

---

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 7 \\ 0 & 0 \end{bmatrix} \quad AB = 0 \text{ but } A \neq 0 \text{ and } B \neq 0$$

---

---

# Identity Matrices (單位矩陣)

- A square matrix with 1's on the main diagonal and 0's off the main diagonal is called an **identity matrix** and is denoted by  $I$ , or  $I_n$  for the  $n \times n$  identity matrix
- If  $A$  is an  $m \times n$  matrix, then  $AI_n = A$  and  $I_m A = A$
- An identity matrix plays the same role in matrix arithmetic as the number 1 plays in the numerical relationships  $a \cdot 1 = 1 \cdot a = a$
- Theorem 1.4.3
  - If  $R$  is the reduced row-echelon form of an  $n \times n$  matrix  $A$ , then either  $R$  has a row of zeros or  $R$  is the identity matrix  $I_n$

---

# Example

- Zero matrices

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad [0]$$

- Identity matrices

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

---

## Proof of Theorem 1.4.3

- Suppose that the reduced row-echelon form of  $A$  is

$$R = \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ r_{21} & r_{22} & \cdots & r_{2n} \\ \vdots & \vdots & & \vdots \\ r_{n1} & r_{n2} & \cdots & r_{nn} \end{bmatrix}$$

- Either the last row in this matrix consists entirely of zeros or it does not.
- If not, the matrix contains no zero rows, and consequently each of the  $n$  rows has a leading entry of 1.

## Proof of Theorem 1.4.3

$$R = \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ r_{21} & r_{22} & \cdots & r_{2n} \\ \vdots & \vdots & & \vdots \\ r_{n1} & r_{n2} & \cdots & r_{nn} \end{bmatrix}$$

- Since these leading 1's occur progressively farther to the right as we move down the matrix, each of these 1's must occur on the main diagonal.
- Since the other entries in the same column as one of these 1's are zero,  $R$  must be  $I_n$ .
- Thus, either  $R$  has a row of zeros or  $R = I_n$ .



---

# Inverse

- If  $A$  is a square matrix, and if a matrix  $B$  of the same size can be found such that  $AB = BA = I$ , then  $A$  is said to be *invertible* (可逆的) or *nonsingular* and  $B$  is called an *inverse* (逆矩陣) of  $A$ . If no such matrix  $B$  can be found, then  $A$  is said to be *singular*. (奇異的)
- Remark:
  - The inverse of  $A$  is denoted as  $A^{-1}$
  - Not every (square) matrix has an inverse
  - An inverse matrix has exactly one inverse

---

# Example

$$B = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} \text{ is an inverse of } A = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}$$

$$AB = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$BA = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$A$  and  $B$  are invertible and each is an inverse of the other.

# Example

The matrix  $A = \begin{bmatrix} 1 & 4 & 0 \\ 2 & 5 & 0 \\ 3 & 6 & 0 \end{bmatrix}$  is singular.

Let  $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$

The third column of  $BA$  is

$$\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow BA \neq I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$j$ th column matrix of  $BA = B[j$ th column matrix of  $A]$

---

# Theorems

- Theorem 1.4.4

- If  $B$  and  $C$  are both inverses of the matrix  $A$ , then  $B = C$

- Theorem 1.4.5

- The matrix  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

is invertible if  $ad - bc \neq 0$ , in which case the inverse is given by the formula

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

- Theorem 1.4.6

- If  $A$  and  $B$  are invertible matrices of the same size, then  $AB$  is invertible and  $(AB)^{-1} = B^{-1}A^{-1}$

---

## Proof of 1.4.4

If  $B$  and  $C$  are both inverses of the matrix  $A$ , then  $B = C$

- Since  $B$  is an inverse of  $A$ , we have  $BA = I$ .
- Multiplying both sides on the right by  $C$  gives  $(BA)C = IC = C$ .
- But  $(BA)C = B(AC) = BI = B$ , so  $C = B$ .

---

## Proof of 1.4.6

If  $A$  and  $B$  are invertible matrices of the same size, then  $AB$  is invertible and  $(AB)^{-1} = B^{-1}A^{-1}$

- If we can show that  $(AB)(B^{-1}A^{-1}) = (B^{-1}A^{-1})(AB) = I$ , then we will have simultaneously shown that the matrix  $AB$  is invertible and that  $(AB)^{-1} = B^{-1}A^{-1}$ .
- $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$ .
- A similar argument shows that  $(B^{-1}A^{-1})(AB) = I$

---

## Example

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix} \quad AB = \begin{bmatrix} 7 & 6 \\ 9 & 8 \end{bmatrix}$$

Applying the formula in Theorem 1.4.5, we obtain

$$A^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} \quad B^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & \frac{3}{2} \end{bmatrix} \quad (AB)^{-1} = \begin{bmatrix} 4 & -3 \\ -\frac{9}{2} & \frac{7}{2} \end{bmatrix}$$

$$B^{-1}A^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & \frac{3}{2} \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ -\frac{9}{2} & \frac{7}{2} \end{bmatrix}$$

---

# Powers of A Matrix

- If  $A$  is a square matrix, then we define the nonnegative integer powers of  $A$  to be

$$A^0 = I \quad A^n = \underbrace{AA \cdots A}_{n \text{ factors}} \quad (n > 0)$$

- If  $A$  is invertible, then we define the negative integer powers to be

$$A^{-n} = (A^{-1})^n = \underbrace{A^{-1}A^{-1} \cdots A^{-1}}_{n \text{ factors}} \quad (n > 0)$$



---

# Example

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \quad A^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 11 & 30 \\ 15 & 41 \end{bmatrix}$$

$$A^{-3} = (A^{-1})^3 = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 41 & -30 \\ -15 & 11 \end{bmatrix}$$

---

# Theorems

- If  $A$  is a square matrix and  $r$  and  $s$  are integers, then  $A^r A^s = A^{r+s}$ ,  $(A^r)^s = A^{rs}$
- Theorem 1.4.7 (Laws of Exponents)
  - If  $A$  is invertible and  $n$  is a nonnegative integer, then:
    - $A^{-1}$  is invertible and  $(A^{-1})^{-1} = A$
    - $A^n$  is invertible and  $(A^n)^{-1} = (A^{-1})^n$  for  $n = 0, 1, 2, \dots$
    - For any nonzero scalar  $k$ , the matrix  $kA$  is invertible and  $(kA)^{-1} = k^{-1}A^{-1} = (1/k)A^{-1}$

---

# Proof

$A^{-1}$  is invertible and  $(A^{-1})^{-1} = A$

- Since  $AA^{-1} = A^{-1}A = I$ , the matrix  $A^{-1}$  is invertible and  $(A^{-1})^{-1} = A$ .

For any nonzero scalar  $k$ , the matrix  $kA$  is invertible and  $(kA)^{-1} = (1/k)A^{-1}$

$$(kA)\left(\frac{1}{k}A^{-1}\right) = \frac{1}{k}(kA)A^{-1} = \left(\frac{1}{k}k\right)AA^{-1} = 1I = I$$

Similarly,  $\left(\frac{1}{k}A^{-1}\right)(kA) = I$

---

# Polynomial Expressions Involving Matrices

- If  $A$  is a square matrix, say  $m \times m$ , and if

$$p(x) = a_0 + a_1x + \dots + a_nx^n$$

is any polynomial, then we define

$$p(A) = a_0I + a_1A + \dots + a_nA^n$$

where  $I$  is the  $m \times m$  identity matrix.

- That is,  $p(A)$  is the  $m \times m$  matrix that results when  $A$  is substituted for  $x$  in the above equation and  $a_0$  is replaced by  $a_0I$

---

## Example (Matrix Polynomial)

If

$$p(x) = 2x^2 - 3x + 4 \quad \text{and} \quad A = \begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix}$$

then

$$\begin{aligned} p(A) &= 2A^2 - 3A + 4I = 2 \begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix}^2 - 3 \begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix} + 4 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 8 \\ 0 & 18 \end{bmatrix} - \begin{bmatrix} -3 & 6 \\ 0 & 9 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 9 & 2 \\ 0 & 13 \end{bmatrix} \end{aligned}$$

---

# Theorems

- Theorem 1.4.8 (Properties of the Transpose)
  - If the sizes of the matrices are such that the stated operations can be performed, then
    - $((A^T)^T = A$
    - $(A + B)^T = A^T + B^T$  and  $(A - B)^T = A^T - B^T$
    - $(kA)^T = kA^T$ , where  $k$  is any scalar
    - $(AB)^T = B^T A^T$
- Theorem 1.4.9 (Invertibility of a Transpose)
  - If  $A$  is an invertible matrix, then  $A^T$  is also invertible and  $(A^T)^{-1} = (A^{-1})^T$

# Proof

If  $A$  is an invertible matrix, then  $A^T$  is also invertible and  $(A^T)^{-1} = (A^{-1})^T$

- We can prove the invertibility of  $A^T$  by showing that

$$A^T(A^{-1})^T = (A^{-1})^T A^T = I$$

$$A^T(A^{-1})^T = (A^{-1}A)^T = I^T = I$$

$$(A^{-1})^T A^T = (AA^{-1})^T = I^T = I$$

$$A = \begin{bmatrix} -5 & -3 \\ 2 & 1 \end{bmatrix} \quad A^T = \begin{bmatrix} -5 & 2 \\ -3 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1 & 3 \\ -2 & -5 \end{bmatrix} \quad (A^{-1})^T = \begin{bmatrix} 1 & -2 \\ 3 & -5 \end{bmatrix} \quad (A^T)^{-1} = \begin{bmatrix} 1 & -2 \\ 3 & -5 \end{bmatrix}$$

---

# 1.5

## Elementary Matrices and a Method for Finding $A^{-1}$

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# Elementary Row Operations

- An **elementary row operation** (sometimes called just a row operation) on a matrix  $A$  is any one of the following three types of operations:
  - Interchange of two rows of  $A$
  - Replacement of a row  $\mathbf{r}$  of  $A$  by  $c\mathbf{r}$  for some number  $c \neq 0$
  - Replacement of a row  $\mathbf{r}_1$  of  $A$  by the sum  $\mathbf{r}_1 + c\mathbf{r}_2$  of that row and a multiple of another row  $\mathbf{r}_2$  of  $A$
- Matrices  $A$  and  $B$  are **row equivalent** if either can be obtained from the other by a sequence of elementary row operations.

---

# Elementary Matrices

- An  $n \times n$  **elementary matrix** (基本矩陣) is a matrix produced by applying exactly one elementary row operation to  $I_n$ 
  - $E_{ij}$  is the elementary matrix obtained by interchanging the  $i$ -th and  $j$ -th rows of  $I_n$
  - $E_i(c)$  is the elementary matrix obtained by multiplying the  $i$ -th row of  $I_n$  by  $c \neq 0$
  - $E_{ij}(c)$  is the elementary matrix obtained by adding  $c$  times the  $j$ -th row to the  $i$ -th row of  $I_n$ , where  $i \neq j$

# Example (Elementary Matrices and Row Operations)

Listed below are four elementary matrices and the operations that produce them.

$$\begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix}$$



Multiply the second row of  $I_2$  by  $-3$ .

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$



Interchange the second and fourth rows of  $I_4$ .

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Add 3 times the third row of  $I_3$  to the first row.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Multiply the first row of  $I_3$  by 1.

---

# Elementary Matrices and Row Operations

- Theorem 1.5.1 (Elementary Matrices and Row Operations)
  - Suppose that  $E$  is an  $m \times m$  elementary matrix produced by applying a particular elementary row operation to  $I_m$ , and that  $A$  is an  $m \times n$  matrix. Then  $EA$  is the matrix that results from applying that same elementary row operation to  $A$
  
- Remark:
  - When a matrix  $A$  is multiplied on the **left** by an elementary matrix  $E$ , the effect is to perform an elementary row operation on  $A$

# Example (Using Elementary Matrices)

Consider the matrix


$$A = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 1 & 4 & 4 & 0 \end{bmatrix}$$

and consider the elementary matrix

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

which results from adding 3 times the first row of  $I_3$  to the third row. The product  $EA$  is

$$EA = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 4 & 4 & 10 & 9 \end{bmatrix}$$

which is precisely the same matrix that results when we add 3 times the first row of  $A$  to the third row. 

---

# Inverse Operations

- If an elementary row operation is applied to an identity matrix  $I$  to produce an elementary matrix  $E$ , then there is a second row operation that, when applied to  $E$ , produces  $I$  back again

# Example

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{\text{Multiply the second row by 7}} \begin{bmatrix} 1 & 0 \\ 0 & 7 \end{bmatrix} \xrightarrow{\text{Multiply the second row by } 1/7} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{\text{Interchange the first and the second rows}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \xrightarrow{\text{Interchange the first and the second rows}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{\text{Add 5 times the second row to the first}} \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix} \xrightarrow{\text{Add -5 times the second row to the first}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

---

## Theorem 1.5.2 (Elementary Matrices and Nonsingularity)

- Each elementary matrix is **nonsingular (is invertible)**, and its inverse is itself an elementary matrix. More precisely,
  - $E_{ij}^{-1} = E_{ji} (= E_{ij})$
  - $E_i(c)^{-1} = E_i(1/c)$  with  $c \neq 0$
  - $E_{ij}(c)^{-1} = E_{ij}(-c)$  with  $i \neq j$



---

## Theorem 1.5.3 (Equivalent Statements)

- If  $A$  is an  $n \times n$  matrix, then the following statements are equivalent, that is, all true or all false
  - (a)  $A$  is invertible
  - (b)  $A\mathbf{x} = \mathbf{0}$  has only the trivial solution
  - (c) The reduced row-echelon form of  $A$  is  $I_n$
  - (d)  $A$  is expressible as a product of elementary matrices

---

# Proof

(a)  $\rightarrow$  (b)

- Assume  $A$  is invertible and let  $\mathbf{x}_0$  be any solution of  $A\mathbf{x} = \mathbf{0}$  thus  $A\mathbf{x}_0 = \mathbf{0}$ .
- Multiplying both sides of this equation by the matrix  $A^{-1}$  gives  $A^{-1}(A\mathbf{x}_0) = A^{-1} \cdot \mathbf{0}$ , or  $(A^{-1}A)\mathbf{x}_0 = \mathbf{0}$ , or  $I\mathbf{x}_0 = \mathbf{0}$ , or  $\mathbf{x}_0 = \mathbf{0}$ .
- Thus,  $A\mathbf{x} = \mathbf{0}$  has only the trivial solution.

---

# Proof

(b)  $\rightarrow$  (c)

- Let  $A\mathbf{x} = \mathbf{0}$  be the matrix form of the system

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0$$

$$\vdots \quad \quad \quad \vdots \quad \quad \quad \vdots \quad \quad \quad \vdots$$

$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = 0$$

- Assume that the system has only the trivial solution. If we solve by Gauss-Jordan elimination, the reduced row-echelon form of the augmented matrix will be

$$x_1 \quad \quad \quad = 0$$

$$x_2 \quad \quad \quad = 0$$

$\cdots$

$$x_n = 0$$

# Proof

(b)  $\rightarrow$  (c)

- The augmented matrix

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & 0 \end{bmatrix}$$

can be reduced to the augmented matrix

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

The reduced row-echelon form of  $A$  is  $I_n$

---

# Proof

(c)  $\rightarrow$  (d)

- Assume that the reduced row-echelon form of  $A$  is  $I_n$ , so that  $A$  can be reduced to  $I_n$  by a finite sequence of elementary row operations.
- By Theorem 1.5.1, each of these operations can be accomplished by multiplying on the left by an appropriate elementary matrix. Thus we can find elementary matrices  $E_1, E_2, \dots, E_k$  such that

$$E_k \cdots E_2 E_1 A = I_n$$

# Proof

$$E_k \cdots E_2 E_1 A = I_n$$

(c)  $\rightarrow$  (d)

- By Theorem 1.5.2,  $E_1, E_2, \dots, E_k$  are invertible. Multiplying both sides on the left successively by  $E_k^{-1}, \dots, E_2^{-1}, E_1^{-1}$  we obtain

$$E_1^{-1} E_2^{-1} \cdots E_k^{-1} (E_k \cdots E_2 E_1 A) = E_1^{-1} E_2^{-1} \cdots E_k^{-1} I_n$$

$$A = E_1^{-1} E_2^{-1} \cdots E_k^{-1} I_n = E_1^{-1} E_2^{-1} \cdots E_k^{-1}$$

- By Theorem 1.5.2, this equation expresses  $A$  as a product of elementary matrices.

---

# A Method for Inverting Matrices

$$E_k \cdots E_2 E_1 A = I_n$$

- Multiplying on the right by  $A^{-1}$  yields

$$E_k \cdots E_2 E_1 A A^{-1} = I_n A^{-1}$$

$$A^{-1} = E_k \cdots E_2 E_1 I_n$$

- $A^{-1}$  can be obtained by multiplying  $I_n$  successively on the left by the elementary matrices  $E_1, E_2, \dots, E_k$ .

- ***The sequence of row operations that reduces  $A$  to  $I_n$  will reduce  $I_n$  to  $A^{-1}$ .***

---

# A Method for Inverting Matrices

- To find the inverse of an invertible matrix  $A$ , we must find a sequence of elementary row operations that reduces  $A$  to the identity and then perform this same sequence of operations on  $I_n$  to obtain  $A^{-1}$

- Remark

- Suppose we can find elementary matrices  $E_1, E_2, \dots, E_k$  such that

$$E_k \dots E_2 E_1 A = I_n$$

then

$$A^{-1} = E_k \dots E_2 E_1 I_n$$



---

## Example (Using Row Operations to Find $A^{-1}$ )

- Find the inverse of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$$

- Solution:
  - To accomplish this we shall adjoin the identity matrix to the right side of  $A$ , thereby producing a matrix of the form  $[A \mid I]$
  - We shall apply row operations to this matrix until the left side is reduced to  $I$ ; these operations will convert the right side to  $A^{-1}$ , so that the final matrix will have the form  $[I \mid A^{-1}]$

# Example

The computations are as follows:

$$\left[ \begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array} \right]$$

$$\left[ \begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array} \right]$$

$$\left[ \begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array} \right]$$

← We added  $-2$  times the first row to the second and  $-1$  times the first row to the third.

← We added 2 times the second row to the third.

# Example

$$\left[ \begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array} \right]$$

← We multiplied the third row by  $-1$ .

$$\left[ \begin{array}{ccc|ccc} 1 & 2 & 0 & -14 & 6 & 3 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array} \right]$$

← We added 3 times the third row to the second and  $-3$  times the third row to the first.

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array} \right]$$

← We added  $-2$  times the second row to the first.

Thus,

$$A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$$

# Example

- Not every matrix is invertible

$$\left[ \begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 2 & 4 & -1 & 0 & 1 & 0 \\ -1 & 2 & 5 & 0 & 0 & 1 \end{array} \right]$$

$$\left[ \begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 0 & -8 & -9 & -2 & 1 & 0 \\ 0 & 8 & 9 & 1 & 0 & 1 \end{array} \right]$$

We added -2 times the first row to the second and added the first row to the third.

$$\left[ \begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 0 & -8 & -9 & -2 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 \end{array} \right]$$

We added the second row to the third.

Since we have obtained a row of zeros on the left side,  $A$  is not invertible.

---

# Example

- Determine whether the given homogeneous system has nontrivial solutions

$$\begin{aligned}x_1 + 2x_2 + 3x_3 &= 0 \\2x_1 + 5x_2 + 3x_3 &= 0 \\x_1 + 8x_3 &= 0\end{aligned}$$

$$\begin{aligned}x_1 + 6x_2 + 4x_3 &= 0 \\2x_1 + 4x_2 - x_3 &= 0 \\-x_1 + 2x_2 + 5x_3 &= 0\end{aligned}$$

$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$  is invertible, and the first system has only trivial solution

$\begin{bmatrix} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 5 \end{bmatrix}$  is not invertible, and the second system has nontrivial solutions

---

# 1.6

## More on Linear Systems and Invertible Matrices

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# Theorems

- Theorem 1.6.1
  - Every system of linear equations has either no solutions, exactly one solution, or in finitely many solutions.
  
- Theorem 1.6.2
  - If  $A$  is an invertible  $n \times n$  matrix, then for each  $n \times 1$  matrix  $\mathbf{b}$ , the system of equations  $A\mathbf{x} = \mathbf{b}$  has exactly one solution, namely,  $\mathbf{x} = A^{-1}\mathbf{b}$ .

---

# Proof of Theorem 1.6.1

- The proof will be complete if we can show that the system has infinitely many solutions if the system has more than one solution.
  - Assume that  $A\mathbf{x} = \mathbf{b}$  has more than one solution, and let  $\mathbf{x}_0 = \mathbf{x}_1 - \mathbf{x}_2$ , where  $\mathbf{x}_1$  and  $\mathbf{x}_2$  are any two distinct solutions. Because  $\mathbf{x}_1$  and  $\mathbf{x}_2$  are distinct,  $\mathbf{x}_0$  is nonzero.
  - $A\mathbf{x}_0 = A(\mathbf{x}_1 - \mathbf{x}_2) = A\mathbf{x}_1 - A\mathbf{x}_2 = \mathbf{b} - \mathbf{b} = \mathbf{0}$
  - $A(\mathbf{x}_1 + k\mathbf{x}_0) = A\mathbf{x}_1 + k(A\mathbf{x}_0) = \mathbf{b} + k\mathbf{0} = \mathbf{b} + \mathbf{0} = \mathbf{b}$ . This says that  $\mathbf{x}_1 + k\mathbf{x}_0$  is a solution of  $A\mathbf{x} = \mathbf{b}$ .
  - Since  $\mathbf{x}_0$  is nonzero and there are infinitely many choices for  $k$ , the system  $A\mathbf{x} = \mathbf{b}$  has infinitely many solutions.
-



# Example

Consider the system of linear equations

$$\begin{aligned}x_1 + 2x_2 + 3x_3 &= 5 \\2x_1 + 5x_2 + 3x_3 &= 3 \\x_1 \quad \quad + 8x_3 &= 17\end{aligned}$$

In matrix form this system can be written as  $A\mathbf{x} = \mathbf{b}$ , where

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 5 \\ 3 \\ 17 \end{bmatrix}$$

In Example 4 of the preceding section we showed that  $A$  is invertible and

$$A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$$

By Theorem 1.6.2 the solution of the system is

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ 17 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

or  $x_1 = 1, x_2 = -1, x_3 = 2$ .

---

# Linear Systems with a Common Coefficient Matrix

- To solve a sequence of linear systems,  $A\mathbf{x} = \mathbf{b}_1$ ,  $A\mathbf{x} = \mathbf{b}_2$ , ...,  $A\mathbf{x} = \mathbf{b}_k$ , with common coefficient matrix  $A$
- If  $A$  is invertible, then the solutions  $\mathbf{x}_1 = A^{-1}\mathbf{b}_1$ ,  $\mathbf{x}_2 = A^{-1}\mathbf{b}_2$ , ...,  $\mathbf{x}_k = A^{-1}\mathbf{b}_k$
- A more efficient method is to form the matrix  $[A|\mathbf{b}_1|\mathbf{b}_2|\dots|\mathbf{b}_k]$
- By reducing it to reduced row-echelon form we can **solve all  $k$  systems at once** by Gauss-Jordan elimination.

# Example

- Solve the systems

$$x_1 + 2x_2 + 3x_3 = 4$$

$$2x_1 + 5x_2 + 3x_3 = 5$$

$$x_1 + 8x_3 = 9$$

$$\left[ \begin{array}{ccc|c|c} 1 & 2 & 3 & 4 & 1 \\ 2 & 5 & 3 & 5 & 6 \\ 1 & 0 & 8 & 9 & -6 \end{array} \right]$$

$$\left[ \begin{array}{ccc|c|c} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & -1 \end{array} \right]$$

$$x_1 + 2x_2 + 3x_3 = 1$$

$$2x_1 + 5x_2 + 3x_3 = 6$$

$$x_1 + 8x_3 = -6$$

$$x_1 = 1, x_2 = 0, x_3 = 1$$

$$x_1 = 2, x_2 = 1, x_3 = -1$$

---

# Theorems

- Theorem 1.6.3

- Let  $A$  be a square matrix

- If  $B$  is a square matrix satisfying  $BA = I$ , then  $B = A^{-1}$

- If  $B$  is a square matrix satisfying  $AB = I$ , then  $B = A^{-1}$

---

## Proof of Theorem 1.6.3

- Assume that  $BA=I$ . If we can show that  $A$  is invertible, the proof can be completed by multiplying  $BA = I$  on both sides by  $A^{-1}$  to obtain

$$BAA^{-1} = IA^{-1} \quad BI = IA^{-1} \quad B = A^{-1}$$

- To show that  $A$  is invertible, it suffices to show that the system  $A\mathbf{x} = \mathbf{0}$  has only the trivial solution.
- Let  $\mathbf{x}_0$  be any solution of this system. If we multiply both sides of  $A\mathbf{x}_0 = \mathbf{0}$  on the left by  $B$ , we obtain  $BA\mathbf{x}_0 = B\mathbf{0}$  or  $I\mathbf{x}_0 = \mathbf{0}$  or  $\mathbf{x}_0 = \mathbf{0}$ . Thus, the system  $A\mathbf{x} = \mathbf{0}$  has only the trivial solution.

---

## Theorem 1.6.4 (Equivalent Statements)

- If  $A$  is an  $n \times n$  matrix, then the following statements are equivalent
  - $A$  is invertible
  - $A\mathbf{x} = \mathbf{0}$  has only the trivial solution
  - The reduced row-echelon form of  $A$  is  $I_n$
  - $A$  is expressible as a product of elementary matrices
  - $A\mathbf{x} = \mathbf{b}$  is consistent for every  $n \times 1$  matrix  $\mathbf{b}$
  - $A\mathbf{x} = \mathbf{b}$  has exactly one solution for every  $n \times 1$  matrix  $\mathbf{b}$

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# Theorems

- Theorem 1.6.5
  - Let  $A$  and  $B$  be square matrices of the same size. If  $AB$  is invertible, then  $A$  and  $B$  must also be invertible.
- *A fundamental problem:* Let  $A$  be a fixed  $m \times n$  matrix. Find all  $m \times 1$  matrices  $\mathbf{b}$  such that the system of equations  $A\mathbf{x} = \mathbf{b}$  is consistent.

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# Fundamental Problem

- If  $A$  is invertible, Theorem 1.6.2 says that  $A\mathbf{x}=\mathbf{b}$  has the unique solution.
- If  $A$  is not square, or if  $A$  is square but not invertible
  - The matrix  $\mathbf{b}$  must usually satisfy certain conditions in order for  $A\mathbf{x}=\mathbf{b}$  to be consistent



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# Example

- What conditions must  $b_1$ ,  $b_2$ , and  $b_3$  satisfy in order for the systems of equations to be consistent?

$$x_1 + 2x_2 + 3x_3 = b_1$$

$$x_1 + x_3 = b_2$$

$$2x_1 + x_2 + 3x_3 = b_3$$

- The augmented matrix is

$$\begin{bmatrix} 1 & 1 & 2 & b_1 \\ 1 & 0 & 1 & b_2 \\ 2 & 1 & 3 & b_3 \end{bmatrix}$$

# Example

$$\begin{bmatrix} 1 & 1 & 2 & b_1 \\ 1 & 0 & 1 & b_2 \\ 2 & 1 & 3 & b_3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 2 & b_1 \\ 0 & -1 & -1 & b_2 - b_1 \\ 0 & -1 & -1 & b_3 - 2b_1 \end{bmatrix}$$

-1 times the first row was added to the second and -2 times the first row was added to the third.

$$\begin{bmatrix} 1 & 1 & 2 & b_1 \\ 0 & 1 & 1 & b_1 - b_2 \\ 0 & -1 & -1 & b_3 - 2b_1 \end{bmatrix}$$

The second row was multiplied by -1.

$$\begin{bmatrix} 1 & 1 & 2 & b_1 \\ 0 & 1 & 1 & b_1 - b_2 \\ 0 & 0 & 0 & b_3 - b_2 - b_1 \end{bmatrix}$$

The second row was multiplied to the third.

## Example

$$\begin{bmatrix} 1 & 1 & 2 & b_1 \\ 0 & 1 & 1 & b_1 - b_2 \\ 0 & 0 & 0 & b_3 - b_2 - b_1 \end{bmatrix}$$

- The system has a solution if and only if  $b_1$ ,  $b_2$ , and  $b_3$  satisfy the condition

$$b_3 - b_2 - b_1 = 0 \quad \text{or} \quad b_3 = b_2 + b_1$$

- To express this condition another way,  $A\mathbf{x}=\mathbf{b}$  is consistent if and only if  $\mathbf{b}$  is a matrix of the form

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_1 + b_2 \end{bmatrix}$$

where  $b_1$  and  $b_2$  are arbitrary.

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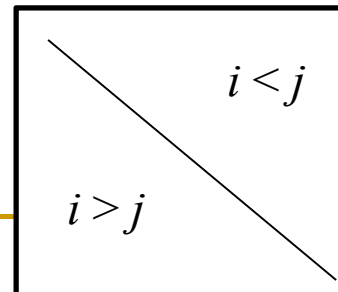
1.7

## Diagonal, Triangular, and Symmetric Matrices

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# Diagonal and Triangular

- A **square matrix**  $A$  is  $m \times n$  with  $m = n$ ; the  $(i,i)$ -entries for  $1 \leq i \leq m$  form the **main diagonal** of  $A$
- A **diagonal matrix** (對角矩陣) is a square matrix all of whose entries *not* on the main diagonal equal zero. By  $\text{diag}(d_1, \dots, d_m)$  is meant the  $m \times m$  diagonal matrix whose  $(i,i)$ -entry equals  $d_i$  for  $1 \leq i \leq m$
- A  $n \times n$  **lower-triangular matrix** (下三角矩陣)  $L$  satisfies  $(L)_{ij} = 0$  if  $i < j$ , for  $1 \leq i \leq m$  and  $1 \leq j \leq n$
- A  $n \times n$  **upper-triangular matrix** (上三角矩陣)  $U$  satisfies  $(U)_{ij} = 0$  if  $i > j$ , for  $1 \leq i \leq m$  and  $1 \leq j \leq n$



# Properties of Diagonal Matrices

- A general  $n \times n$  diagonal matrix  $D$  can be written as

$$D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}$$

- A diagonal matrix is invertible if and only if all of its diagonal entries are nonzero

$$D^{-1} = \begin{bmatrix} 1/d_1 & 0 & \cdots & 0 \\ 0 & 1/d_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1/d_n \end{bmatrix}$$

- Powers of diagonal matrices are easy to compute

$$D^k = \begin{bmatrix} d_1^k & 0 & \cdots & 0 \\ 0 & d_2^k & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_n^k \end{bmatrix}$$

# Example

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

$$A^5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -243 & 0 \\ 0 & 0 & 32 \end{bmatrix}$$

$$A^{-5} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{243} & 0 \\ 0 & 0 & \frac{1}{32} \end{bmatrix}$$

# Properties of Diagonal Matrices

- Matrix products that involve diagonal factors are especially easy to compute

$$\begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} = \begin{bmatrix} d_1 a_{11} & d_1 a_{12} & d_1 a_{13} & d_1 a_{14} \\ d_2 a_{21} & d_2 a_{22} & d_2 a_{23} & d_2 a_{24} \\ d_3 a_{31} & d_3 a_{32} & d_3 a_{33} & d_3 a_{34} \end{bmatrix}$$
$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix} \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} = \begin{bmatrix} d_1 a_{11} & d_2 a_{12} & d_3 a_{13} \\ d_1 a_{21} & d_2 a_{22} & d_3 a_{23} \\ d_1 a_{31} & d_2 a_{32} & d_3 a_{33} \\ d_1 a_{41} & d_2 a_{42} & d_3 a_{43} \end{bmatrix}$$

To multiply a matrix  $A$  on the left by a diagonal matrix  $D$ , one can multiply successive rows of  $A$  by the successive diagonal entries of  $D$ .

To multiply  $A$  on the right by  $D$ , one can multiply successive columns of  $A$  by the successive diagonal entries of  $D$ .



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## Theorem 1.7.1

- The transpose of a lower triangular matrix is upper triangular, and the transpose of an upper triangular matrix is lower triangular
- The product of lower triangular matrices is lower triangular, and the product of upper triangular matrices is upper triangular
- A triangular matrix is invertible if and only if its diagonal entries are all nonzero
- The inverse of an invertible lower triangular matrix is lower triangular, and the inverse of an invertible upper triangular matrix is upper triangular

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## Example

$$A = \begin{bmatrix} 1 & 3 & -1 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix} \quad B = \begin{bmatrix} 3 & -2 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

The matrix  $A$  is invertible, since its diagonal entries are nonzero, but the matrix  $B$  is not.

$$A^{-1} = \begin{bmatrix} 1 & -\frac{3}{2} & \frac{7}{5} \\ 0 & \frac{1}{2} & -\frac{2}{5} \\ 0 & 0 & \frac{1}{5} \end{bmatrix}$$

The product  $AB$  is also upper triangular.

$$AB = \begin{bmatrix} 3 & -2 & -2 \\ 0 & 0 & 2 \\ 0 & 0 & 5 \end{bmatrix}$$

# Proof

The product of lower triangular matrices is lower triangular.

- Let  $A = [a_{ij}]$  and  $B = [b_{ij}]$  be lower triangular  $n \times n$  matrices, and let  $C = [c_{ij}]$  be the product  $C=AB$ .
- We can prove that  $C$  is lower triangular by showing that  $c_{ij} = 0$  for  $i < j$ .

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

- If we assume that  $i < j$ , then the terms can be grouped as

$$c_{ij} = \underbrace{a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{i(j-1)}b_{(j-1)j}}_{\text{The row number of } b \text{ is less than the column number of } b} + \underbrace{a_{ij}b_{jj} + \cdots + a_{in}b_{nj}}_{\text{The row number of } a \text{ is less than the column number of } a}$$

The row number of  $b$  is less than the column number of  $b$

The row number of  $a$  is less than the column number of  $a$

# Proof

The product of lower triangular matrices is lower triangular.

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{i(j-1)}b_{(j-1)j} + a_{ij}b_{jj} + \cdots + a_{in}b_{nj}$$

The row number of  $b$  is less than the column number of  $b$

The row number of  $a$  is less than the column number of  $a$

- In the first grouping all of the  $b$  factors are zero since  $B$  is lower triangular. In the second grouping all of the  $a$  factors are zero since  $A$  is lower triangular. Thus,  $c_{ij}=0$ .

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# Symmetric Matrices

- Definition

- A (square) matrix  $A$  for which  $A^T = A$ , so that  $\langle A \rangle_{ij} = \langle A \rangle_{ji}$  for all  $i$  and  $j$ , is said to be **symmetric**.

- Theorem 1.7.2

- If  $A$  and  $B$  are symmetric matrices (對稱矩陣) with the same size, and if  $k$  is any scalar, then
  - $A^T$  is symmetric
  - $A + B$  and  $A - B$  are symmetric
  - $kA$  is symmetric

- Theorem 1.7.3

- The product of two symmetric matrices is symmetric if and only if the matrices **commute** (可交換), i.e.,  $AB = BA$

# Example

- It is not true, in general, that the product of symmetric matrices is symmetric.

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -4 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -5 & 2 \end{bmatrix}$$

- If these two matrices commute, the product of two symmetric matrices is symmetric.

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -4 & 3 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \quad \begin{bmatrix} -4 & 3 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$$

---

# Theorems

- Theorem 1.7.4

- If  $A$  is an invertible symmetric matrix, then  $A^{-1}$  is symmetric.

- Remark:

- In general, a symmetric matrix needs not be invertible.
- The products  $AA^T$  and  $A^T A$  are always symmetric

- Theorem 1.7.5

- If  $A$  is an invertible matrix, then  $AA^T$  and  $A^T A$  are also invertible

---

# Proof

If  $A$  is an invertible symmetric matrix, then  $A^{-1}$  is symmetric.

- Assume that  $A$  is symmetric and invertible. From Theorem 1.4.9 and the fact that  $A=A^T$ , we have

$$(A^{-1})^T = (A^T)^{-1} = A^{-1}$$

which proves that  $A^{-1}$  is symmetric.

Theorem 1.4.9

$$(A^{-1})^T = (A^T)^{-1}$$

The products  $AA^T$  and  $A^T A$  are always symmetric

- $(AA^T)^T = (A^T)^T A^T = AA^T$
- $(A^T A)^T = A^T (A^T)^T = A^T A$



# Example

Let  $A$  be the  $2 \times 3$  matrix

$$A = \begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & -5 \end{bmatrix}$$

Then

$$A^T A = \begin{bmatrix} 1 & 3 \\ -2 & 0 \\ 4 & -5 \end{bmatrix} \begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & -5 \end{bmatrix} = \begin{bmatrix} 10 & -2 & -11 \\ -2 & 4 & -8 \\ -11 & -8 & 41 \end{bmatrix}$$
$$A A^T = \begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & -5 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -2 & 0 \\ 4 & -5 \end{bmatrix} = \begin{bmatrix} 21 & -17 \\ -17 & 34 \end{bmatrix}$$

Observe that  $A^T A$  and  $A A^T$  are symmetric as expected.

---

# Proof

If  $A$  is an invertible matrix, then  $AA^T$  and  $A^T A$  are also invertible

- Since  $A$  is invertible, so is  $A^T$  by Theorem 1.4.9.
- Thus  $AA^T$  and  $A^T A$  are invertible, since they are the products of invertible matrices.

Theorem 1.4.9

$$(A^{-1})^T = (A^T)^{-1}$$