

# Clustering Techniques for Information Retrieval

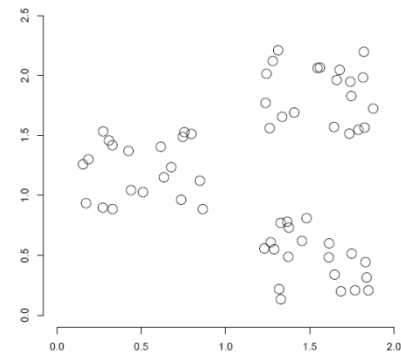
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## References:

1. Christopher D. Manning, Prabhakar Raghavan and Hinrich Schütze, Introduction to Information Retrieval, Cambridge University Press, 2008. (Chapters 16 & 17)
2. *Modern Information Retrieval*, Chapters 5 & 7
3. "A Gentle Tutorial of the EM Algorithm and its Application to Parameter Estimation for Gaussian Mixture and Hidden Markov Models," Jeff A. Bilmes, U.C. Berkeley TR-97-021

# Clustering



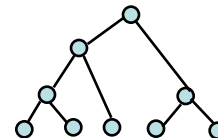
- Place **similar objects** in the same group and assign **dissimilar objects** to different groups (typically using a distance measure, such as Euclidean distance)
  - **Word clustering**
    - **Neighbor overlap**: words occur with the similar left and right neighbors (such as *in* and *on*)
  - **Document clustering**
    - Documents with the similar topics or concepts are put together
- Nevertheless, clustering cannot give a comprehensive description of the object
  - How to **label** objects shown on the visual display is a difficult problem

# Clustering vs. Classification

- Classification is **supervised** and requires a set of labeled training instances for each group (class)
  - Learning with a teacher
- Clustering is **unsupervised** and learns without a teacher to provide the labeling information of the training data set
  - Also called automatic or unsupervised classification

# Types of Clustering Algorithms

- Two types of structures produced by clustering algorithms
  - Flat or non-hierarchical clustering
  - Hierarchical clustering
- **Flat clustering**
  - Simply consisting of a certain number of clusters and the relation between clusters is often undetermined
  - **Measurement:** construction error minimization or probabilistic optimization
- **Hierarchical clustering**
  - A hierarchy with usual interpretation that each node stands for a sub-cluster of its mother's node
    - The leaves of the tree are the single objects
    - Each node represents the cluster that contains all the objects of its descendants
  - **Measurement:** similarities of instances



# Hard Assignment vs. Soft Assignment (1/2)

- Another important distinction between clustering algorithms is whether they perform soft or hard assignment
- **Hard Assignment**
  - Each object (or document in the context of IR) is assigned to one and only one cluster
- **Soft Assignment (probabilistic approach)**
  - Each object may be assigned to multiple clusters
  - An object  $x_i$  has a probability distribution  $P(\cdot|x_i)$  over clusters  $c_j$  where  $P(x_i|c_j)$  is the probability that  $x_i$  is a member of  $c_j$
  - Is somewhat more appropriate in many tasks such as NLP, IR, ...

## Hard Assignment vs. Soft Assignment (2/2)

- Hierarchical clustering usually adopts hard assignment
- While in flat clustering, both types of assignments are common

# Summarized Attributes of Clustering Algorithms (1/2)

- Hierarchical Clustering
  - Preferable for detailed data analysis
  - Provide more information than flat clustering
  - No single best algorithm (each of the algorithms is seemingly only applicable/optimal for some applications)
  - Less efficient than flat clustering (minimally have to compute  $n \times n$  matrix of similarity coefficients)

# Summarized Attributes of Clustering Algorithms (2/2)

- Flat Clustering

- Preferable if efficiency is a consideration or data sets are very large
- *K*-means is the conceptually feasible method and should probably be used on a new data because its results are often sufficient
  - *K*-means assumes a simple Euclidean representation space, and so cannot be used for many data sets, e.g., nominal data like colors (or samples with features of different scales)
- The EM algorithm is the most choice. It can accommodate definition of clusters and allocation of objects based on complex probabilistic models
  - Its extensions can be used to handle topological/hierarchical orders of samples
    - E.g., Probabilistic Latent Semantic Analysis (PLSA)



# Some Applications of Clustering in IR (1/5)

- **Cluster Hypothesis** (for IR): Documents in the same cluster behave similarly with respect to relevance to information needs
- Possible applications of Clustering in IR

Application	What is clustered?	Benefit	Example
Result set clustering	result set	more effective information presentation to user	Figure 16.2
Scatter-Gather	(subsets of) collection	alternative user interface: “search without typing”	Figure 16.3
Collection clustering	collection	effective information presentation for exploratory browsing	McKeown et al. (2002), <a href="http://news.google.com">http://news.google.com</a>
Language modeling	collection	increased precision and/or recall	Liu and Croft (2004)
Cluster-based retrieval	collection	higher efficiency: faster search	Salton (1971a)

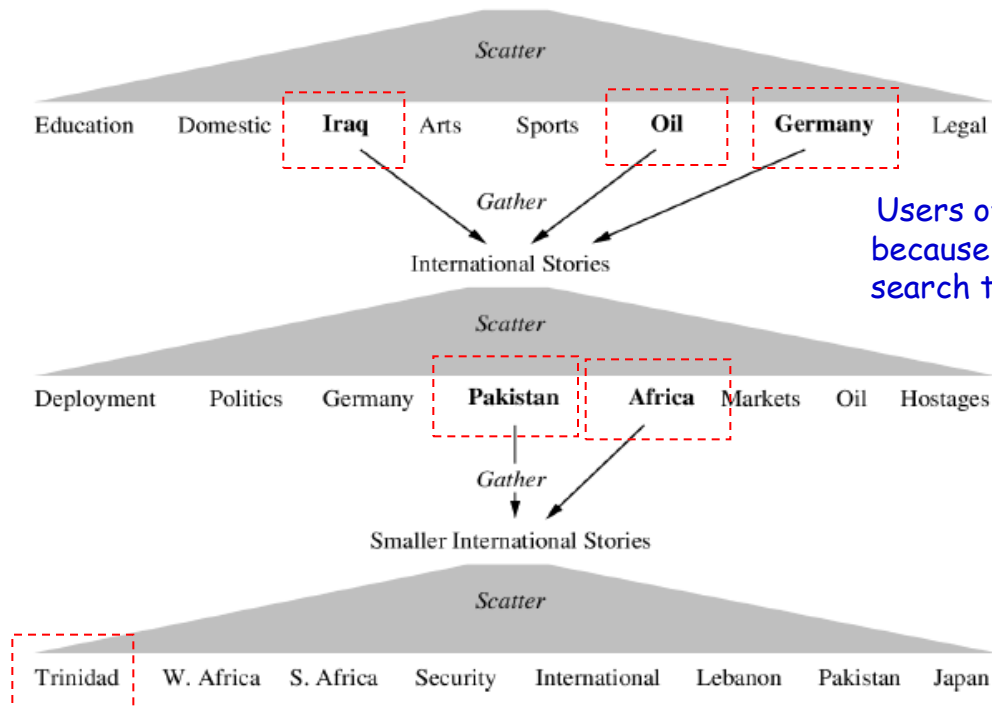
– **These possible applications differ in**

- The collection of documents to be clustered
- The aspect of the IR system to be improved

# Some Applications of Clustering in IR (2/5)

## 1. Whole corpus analysis/navigation

- Better user interface (users prefer browsing over searching since they are unsure about which search terms to use)
- E.g., the *scatter-gather* approach (for a collection of New York Times)



Users often prefer browsing over searching, because they are unsure about which search terms to use.

► **Figure 16.3** The Scatter-Gather user interface. A collection of New York Times news stories is clustered ("scattered") into eight clusters (top row). The user manually *gathers* three of these into a smaller collection *International Stories* and performs another scattering operation. This process repeats until a small cluster with relevant documents is found (e.g., *Trinidad*).

# Some Applications of Clustering in IR (3/5)

## 2. Improve recall in search applications


- Achieve better search results by
  - Alleviating the term-mismatch (synonym) problem facing **the vector space model**
    - First, identify an initial set of documents that match the query (i.e., contain some of the query words)
    - Then, add other documents from the same clusters even if they have low similarity to the query
  - Estimating the collection model of the **language modeling (LM) retrieval approach** more accurately



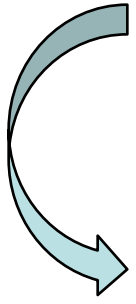
A diagram showing a central blue circle surrounded by several white circles, with a red arrow pointing from the center to one of the white circles. Below it, the text "found relevant document" is written in green.

$$P(Q|M_D) = \prod_{i=1}^N [\lambda \cdot P(w_i|M_D) + (1 - \lambda) \cdot P(w_i|M_C)]$$

The collection model can be estimated from the cluster the document  $D$  belongs to, instead of the entire collection



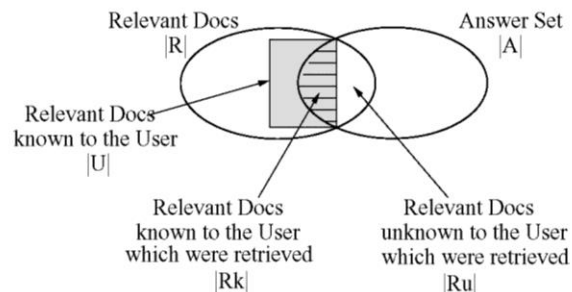
A diagram showing a central blue circle surrounded by several white circles, with a red arrow pointing from the center to one of the white circles. Below it, the text "found relevant document" is written in green.


$$\hat{P}(Q|M_D) = \prod_{i=1}^N [\lambda \cdot P(w_i|M_D) + (1 - \lambda) \cdot P(w_i|M_{Cluster(D)})]$$

# Some Applications of Clustering in IR (4/5)

## 3. Better navigation of search results

- Result set clustering
- Effective “user recall” will be higher



The screenshot shows the Clusty search engine interface. At the top, there is a navigation bar with links for 'web', 'news', 'images', 'wikipedia', 'blogs', 'jobs', and 'more'. The search bar contains the query 'jaguar' and a 'Search' button. Below the search bar, there are tabs for 'clusters', 'sources', and 'sites'. The 'clusters' tab is active, showing a list of clusters for 'All Results (235)'. The clusters include 'Jaguar Cars (33)', 'Parts (33)', 'Photos (30)', 'Jacksonville (23)', 'Club (29)', 'Onca, Panthera (12)', 'X-Type (8)', 'Land Rover (8)', 'Mac OS X (7)', and 'Highlights (5)'. Below the clusters, there is a search box for finding items within the clusters and a 'Find' button. The main search results area shows the top 235 results for the query 'jaguar'. The first result is 'Jaguar' with a Wikipedia snippet: 'The jaguar (*Panthera onca*) is a large member of the cat family native to warm regions of the Americas. It is closely related to the lion, tiger, and leopard of the Old World, and is the largest species of the cat family found in the Americas. en.wikipedia.org/wiki/Jaguar - [cache] - Wikipedia, Live, Ask'. The second result is 'Jaguar' with a snippet: 'Official worldwide web site of Jaguar Cars. Directs users to pages tailored to country-specific markets. www.jaguar.com - [cache] - Live, Open Directory, Ask'. The third result is 'Jag-lovers' with a snippet: 'All Jaguar's Cars. We support our users by hosting multiple Web Sites and Web-based Forums for the various Jaguar models ... are registered trademarks and are the property of Jaguar Cars, England. Some images may also be © Jaguar Cars. Mirroring ... www.jag-lovers.org - [cache] - Gigablast, Open Directory, Ask'. The fourth result is 'Jacksonville Jaguars' with a snippet: 'The official team site with scores, news items, game schedule, and roster. www.jaguars.com - [cache] - Live, Open Directory'. The fifth result is 'www.jaguarusa.com' with a snippet: 'Build Your Jaguar. Request Brochure. Get Email Updates. Locate a Dealer. Search. Your Profile. Site Map. Contact Us. ... - XK. XJ. S-TYPE. X-TYPE. PRE-OWNED. LATEST. OWNERSHIP. Highlights.'

# Some Applications of Clustering in IR (5/5)

## 4. Speed up the search process

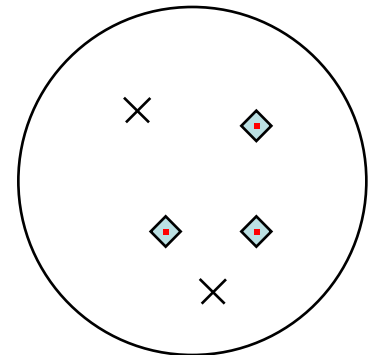
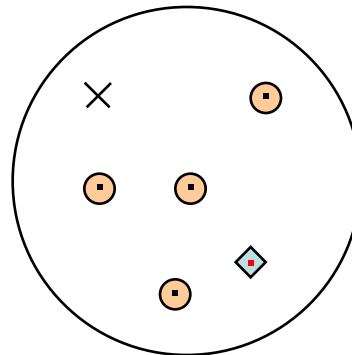
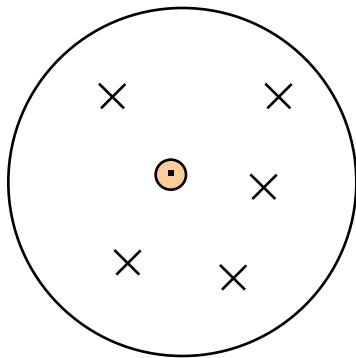
- For retrieval models using exhaustive matching (computing the similarity of the query to every document) without efficient inverted index supports
  - E.g., latent semantic analysis (LSA), language modeling (LM) ?
- Solution: cluster-based retrieval
  - First find the clusters that are closet to the query and then only consider documents from these clusters
    - Within this much smaller set, we can compute similarities exhaustively and rank documents in the usual way

# Evaluation of Clustering (1/2)

- **Internal criterion** for the quality of a clustering result
  - The typical objective is to attain
    - High intra-cluster similarity (documents within a cluster are similar)
    - Low inter-cluster similarity (documents from different clusters are dissimilar)
  - The measured quality depends on both the document representation and the similarity measure used
  - **Good scores on an internal criterion do not necessarily translate into good effectiveness in an application**

# Evaluation of Clustering (2/2)

- **External criterion** for the quality of a clustering result
  - Evaluate how well the clustering matches the **gold standard** classes produced by human judges
    - That is, the quality is measured by the ability of the clustering algorithm to discover some or all of the hidden patterns or latent (true) classes



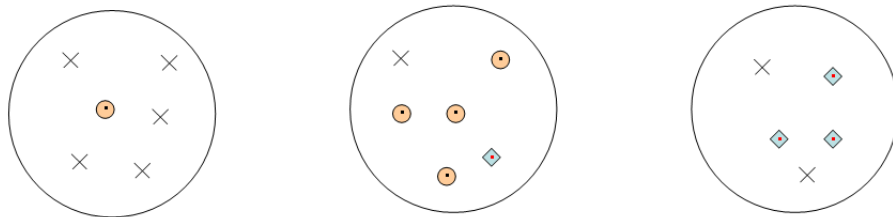
- Two common criteria
  - Purity
  - Rand Index (RI)

# Purity (1/2)

- Each cluster is first assigned to class which is most frequent in the cluster
- Then, the accuracy of the assignment is measured by counting the number of correctly assigned documents and dividing by the sample size

$$\text{Purity}(\Omega, \Gamma) = \frac{1}{N} \sum_k \max_j |\omega_j \cap c_k|$$

- $\Omega = \{\omega_1, \omega_2, \dots, \omega_K\}$ : the set of clusters
- $\Gamma = \{c_1, c_2, \dots, c_J\}$ : the set of classes
- $N$ : the sample size



$$\text{Purity}(\Omega, \Gamma) = \frac{1}{17} (5 + 4 + 3) = 0.71$$



# Purity (2/2)

- High purity is easy to achieve for a large number of clusters (?)
  - Purity will be 1 if each document gets its own cluster
  - Therefore, purity cannot be used to trade off the quality of the clustering against the number of clusters

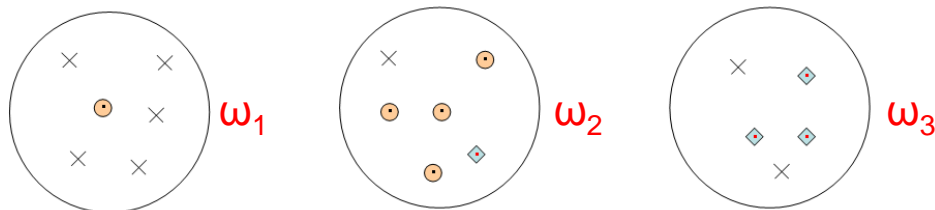
# Rand Index (1/3)

- Measure the similarity between the clusters and the classes in **ground truth**
  - Consider the assignments of all possible  $N(N-1)/2$  pairs of  $N$  distinct documents in the cluster and the true class

<b>Number of document pairs</b>	Same cluster in clustering	Different clusters in clustering
Same class in ground truth	<b>TP</b> <b>(True Positive)</b>	<b>FN</b> <b>(False Negative)</b>
Different classes in ground truth	<b>FP</b> <b>(False Positive)</b>	<b>TN</b> <b>(True Negative)</b>

$$\text{RI} = \frac{\text{TP} + \text{TN}}{\text{TP} + \text{FP} + \text{FN} + \text{TN}}$$

# Rand Index (2/3)



$$TP = \binom{5}{2} + \binom{4}{2} + \left[ \binom{3}{2} + \binom{2}{2} \right] = 20$$

$\omega_1$     $\omega_2$     $\omega_3$

$$FP = \binom{5}{1} \binom{1}{1} + \left[ \binom{4}{1} \binom{1}{1} + \binom{4}{1} \binom{1}{1} + \binom{1}{1} \binom{1}{1} \right] + \binom{3}{1} \binom{2}{1} = 20$$

$\omega_1$     $\omega_2$     $\omega_3$

$$\left( = \left[ \binom{6}{2} + \binom{6}{2} + \binom{5}{2} \right] - TP \right)$$

$\omega_1$     $\omega_2$     $\omega_3$

all positive pairs

	Same cluster	Different clusters
Same class	TP=20	FN=24
Different classes	FP=20	TN=72

$$TN = \left[ \binom{5}{1} \binom{4}{1} + \binom{5}{1} \binom{1}{1} + \binom{1}{1} \binom{1}{1} + \binom{1}{1} \binom{1}{1} \right] + \left[ \binom{5}{1} \binom{3}{1} + \binom{1}{1} \binom{3}{1} + \binom{1}{1} \binom{2}{1} \right] + \left[ \binom{4}{1} \binom{3}{1} + \binom{4}{1} \binom{2}{1} + \binom{1}{1} \binom{3}{1} + \binom{1}{1} \binom{2}{1} \right] = 72$$

$\omega_1\omega_2$     $\omega_1\omega_3$     $\omega_2\omega_3$

$$FN = \left[ \binom{5}{1} \binom{1}{1} + \binom{1}{1} \binom{4}{1} \right] + \left[ \binom{5}{1} \binom{2}{1} + \left[ \binom{1}{1} \binom{2}{1} + \binom{1}{1} \binom{3}{1} \right] \right] = 24$$

$\omega_1\omega_2$     $\omega_1\omega_3$     $\omega_2\omega_3$

$$\left( = \left[ \binom{6}{1} \binom{6}{1} + \binom{6}{1} \binom{5}{1} + \binom{6}{1} \binom{5}{1} \right] - TN \right)$$

$\omega_1\omega_2$     $\omega_1\omega_3$     $\omega_2\omega_3$

all negative pairs

$$RI = \frac{20 + 72}{20 + 20 + 24 + 72} = 0.68$$

all pairs

$$N(N-1)/2 = 17 \times 16 / 2 = 136$$

# Rand Index (3/3)

- The rand index has a value between 0 and 1
  - 0 indicates that the clusters and the classes in ground truth do not agree on any pair of points (documents)
  - 1 indicates that the clusters and the classes in ground truth are exactly the same

# F-Measure Based on Rand Index

- F-Measure: harmonic mean of precision (P) and recall (R)

$$P = \frac{TP}{TP + FP}, \quad R = \frac{TP}{TP + FN}$$

$$F_b = \frac{b^2 + 1}{\frac{b^2}{R} + \frac{1}{P}} = \frac{(b^2 + 1)PR}{b^2 P + R}$$

	Same cluster	Different clusters
Same class	TP	FN
Different classes	FP	TN

- If we want to penalize false negatives (FN) more strongly than false positives (FP), then we can set  $b > 1$  (separating similar documents is sometimes worse than putting dissimilar documents in the same cluster)
  - That is, giving more weight to recall (R)

# Normalized Mutual Information (NMI)

- NMI is an information-theoretical measure

$$\text{NMI}(\Omega, C) = \frac{I(\Omega; C)}{(H(\Omega) + H(C)) / 2}$$

$$\begin{aligned} I(\Omega; C) &= \sum_k \sum_j p(\omega_k \cap c_j) \log \frac{p(\omega_k \cap c_j)}{p(\omega_k)p(c_j)} \\ &= \sum_k \sum_j \frac{|\omega_k \cap c_j|}{N} \log \frac{N|\omega_k \cap c_j|}{|\omega_k||c_j|} \quad (\text{MLEstimate}) \end{aligned}$$

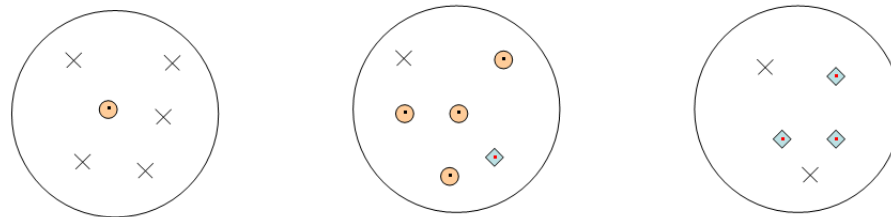
$$\begin{aligned} H(\Omega) &= -\sum_k p(\omega_k) \log p(\omega_k) \\ &= -\sum_k \frac{|\omega_k|}{N} \log \frac{|\omega_k|}{N} \quad (\text{MLEstimate}) \end{aligned}$$

- NMI will have a value between 0 and 1
- NMI has the same problem as purity
  - NMI does not penalize large cardinalities and thus does not formalize our bias, other thing being equal, fewer clusters are better

# Summary of External Evaluation Measures

**Table 16.2** The four external evaluation measures applied to the clustering in Figure 16.4.

	purity	NMI	RI	$F_5$
lower bound	0.0	0.0	0.0	0.0
maximum	1.0	1.0	1.0	1.0
value for Figure 16.4	0.71	0.36	0.68	0.46



# Flat Clustering



# Flat Clustering

- Start out with a partition based on **randomly selected seeds** (one seed per cluster) and then refine the initial partition
  - In a multi-pass manner (recursion/iterations)
- **Problems** associated with non-hierarchical clustering
  - When to stop ? *group average similarity, likelihood, mutual information*
  - What is the right number of clusters (cluster cardinality) ?  
*k-1 → k → k+1*
- **Algorithms** introduced here
  - The *K*-means algorithm
  - The EM algorithm

*Hierarchical clustering is also faced with this problem*

# The $K$ -means Algorithm (1/10)

- Also called *Linde-Buzo-Gray* (LBG) in signal processing
  - A **hard clustering** algorithm
  - Define clusters by the **center of mass** of their members
  - Objects (e.g., documents) should be represented in **vector form**
- The  $K$ -means algorithm also can be regarded as
  - A kind of vector quantization
    - Map from a continuous space (high resolution) to a discrete space (low resolution)
  - E.g. color quantization
    - 24 bits/pixel (16 million colors)  $\rightarrow$  8 bits/pixel (256 colors)
    - A compression rate of 3

$$\mathbf{X} = \left\{ \mathbf{x}^t \right\}_{t=1}^n \xrightarrow{\text{index } j} \mathbf{F} = \left\{ \mathbf{m}_j \right\}_{j=1}^k \quad \text{Dim}(\mathbf{x}^t)=24 \rightarrow |\mathcal{F}|=2^8$$

$\mathbf{m}_j$  : cluster centroid or reference vector, code word, code vector

# The $K$ -means Algorithm (2/10)

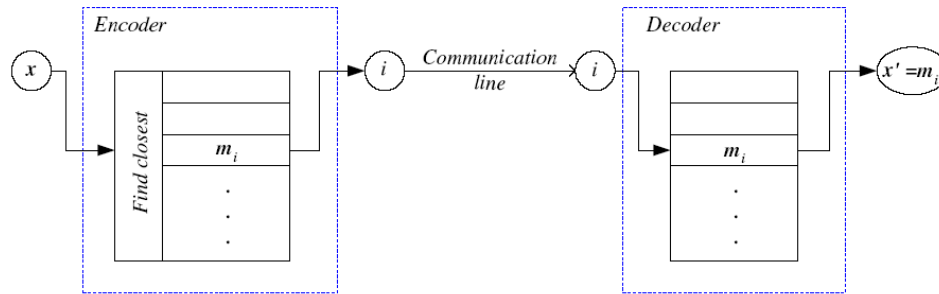


Figure 7.1: Given  $\mathbf{x}$ , the encoder sends the index of the closest code word and the decoder generates the code word with the received index as  $\mathbf{x}'$ . Error is  $\|\mathbf{x}' - \mathbf{x}\|^2$ .

Total reconstruction error (RSS : residual sum of squares)  
automatic label

$$E(\{\mathbf{m}_i\}_{i=1}^k | \mathbf{X}) = \sum_{t=1}^N \sum_{i=1}^k b_i^t \|\mathbf{x}^t - \mathbf{m}_i\|^2, \text{ where } b_i^t = \begin{cases} 1 & \text{if } \|\mathbf{x}^t - \mathbf{m}_i\| = \min_j \|\mathbf{x}^t - \mathbf{m}_j\| \\ 0 & \text{otherwise} \end{cases}$$

- $b_i^t$  and  $m_i$  are unknown in advance
- $b_i^t$  depends on  $m_i$  and this optimization problem can not be solved analytically

# The $K$ -means Algorithm (3/10)

- **Initialization**

- A set of initial cluster centers is needed  $\{\mathbf{m}_i\}_{i=1}^k$

- **Recursion**

- Assign each object  $\mathbf{x}^t$  to the cluster whose center is closest

$$b_i^t = \begin{cases} 1 & \text{if } \|\mathbf{x}^t - \mathbf{m}_i\| = \min_j \|\mathbf{x}^t - \mathbf{m}_j\| \\ 0 & \text{otherwise} \end{cases}$$

- Then, re-compute the center of each cluster as the **centroid** or mean (average) of its members

$$\mathbf{m}_i = \frac{\sum_{t=1}^N b_i^t \cdot \mathbf{x}^t}{\sum_{t=1}^N b_i^t}$$

These two steps are repeated until  $\mathbf{m}_i$  stabilizes (a stopping criterion)

- Or, we can instead use the **medoid** as the cluster center ? (a medoid is one of the objects in the cluster that is closest to the centroid)

# The $K$ -means Algorithm (4/10)

- Algorithm

Initialize  $\mathbf{m}_i, i = 1, \dots, k$ , for example, to  $k$  random  $\mathbf{x}^t$

Repeat

For all  $\mathbf{x}^t \in \mathcal{X}$

$$b_i^t \leftarrow \begin{cases} 1 & \text{if } \|\mathbf{x}^t - \mathbf{m}_i\| = \min_j \|\mathbf{x}^t - \mathbf{m}_j\| \\ 0 & \text{otherwise} \end{cases}$$

For all  $\mathbf{m}_i, i = 1, \dots, k$

$$\mathbf{m}_i \leftarrow \sum_t b_i^t \mathbf{x}^t / \sum_t b_i^t$$

Until  $\mathbf{m}_i$  converge

# The $K$ -means Algorithm (5/10)

- Example 1

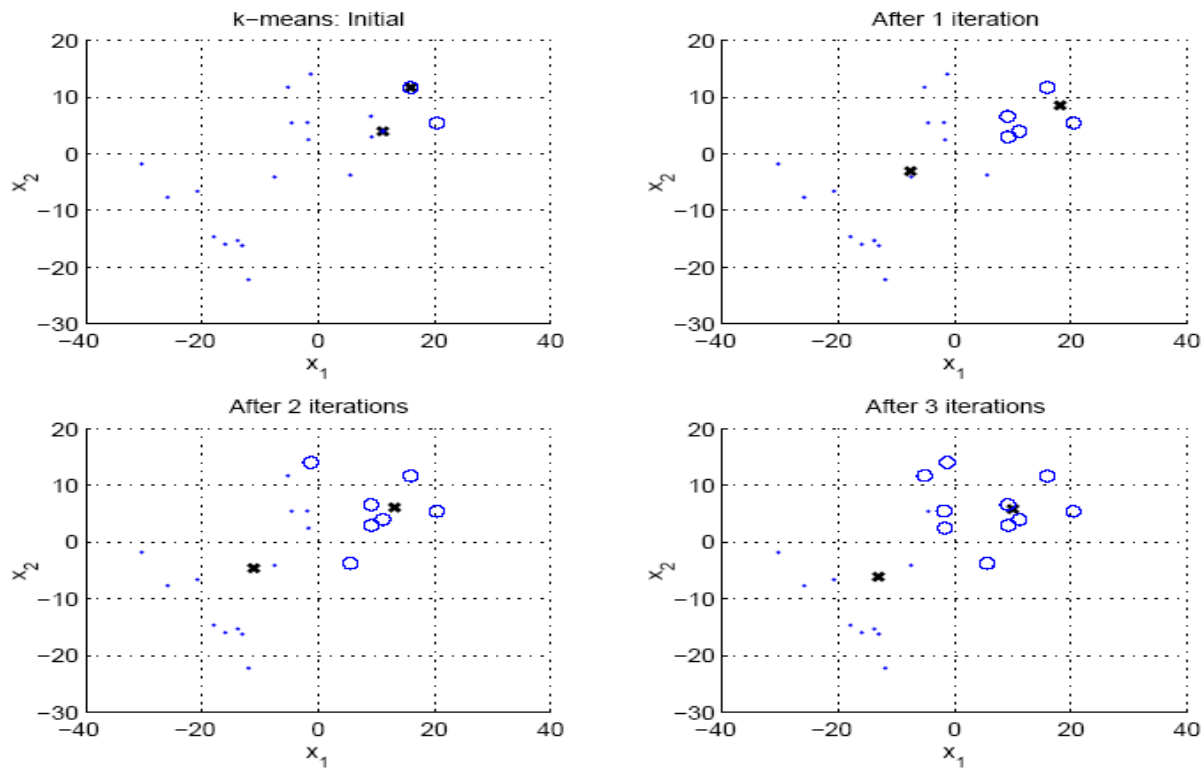


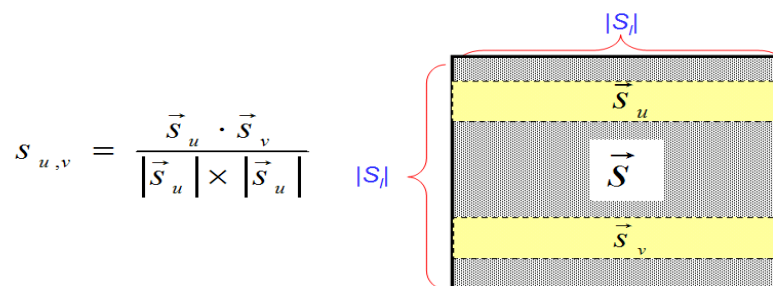
Figure 7.2: Evolution of  $k$ -means. Crosses indicate center positions. Data points are marked depending on the closest center.

# The *K*-means Algorithm (6/10)

- Example 2

Cluster	Members	
1	<i>ballot</i> (0.28), <i>polls</i> (0.28), <i>Gov</i> (0.30), <i>seats</i> (0.32)	government
2	<i>profit</i> (0.21), <i>finance</i> (0.21), <i>payments</i> (0.22)	finance
3	<i>NFL</i> (0.36), <i>Reds</i> (0.28), <i>Sox</i> (0.31), <i>inning</i> (0.33), <i>quarterback</i> (0.30), <i>scored</i> (0.30), <i>score</i> (0.33)	sports
4	<i>researchers</i> (0.23), <i>science</i> (0.23)	research
5	<i>Scott</i> (0.28), <i>Mary</i> (0.27), <i>Barbara</i> (0.27), <i>Edward</i> (0.29)	name

**Table 14.4** An example of K-means clustering. Twenty words represented as vectors of co-occurrence counts were clustered into 5 clusters using K-means. The distance from the cluster centroid is given after each word.



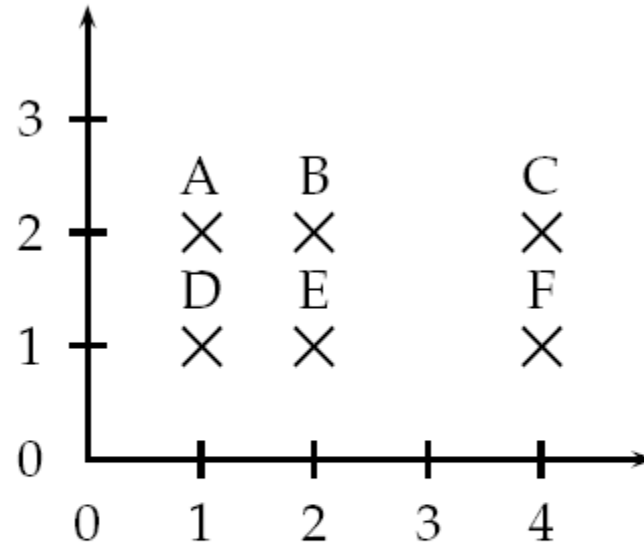
# The $K$ -means Algorithm (7/10)

- Complexity:  $O(IKNM)$ 
  - $I$ : Iterations;  $K$ : cluster number;  $N$ : object number;  $M$ : object dimensionality
- Choice of initial cluster centers (seeds) is important
  - Pick at random
  - Or, calculate the mean  $\mathbf{m}$  of all data and generate  $k$  initial centers  $\mathbf{m}_i$  by adding small random vector to the mean  $\mathbf{m} \pm \delta$
  - Or, project data onto the **principal component** (first eigenvector), divide its range into  $k$  equal intervals, and take the mean of data in each group as the initial center  $\mathbf{m}_i$
  - Or, use another method such as hierarchical clustering algorithm on a subset of the objects
    - E.g., **buckshot algorithm** uses the group-average agglomerative clustering to randomly sample of the data that has size square root of the complete set



# The $K$ -means Algorithm (8/10)

- Poor seeds will result in **sub-optimal** clustering

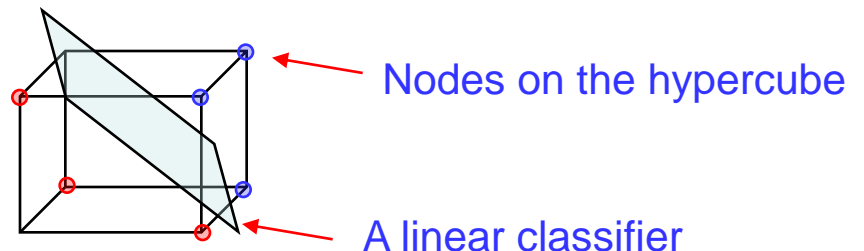


► **Figure 16.7** The outcome of clustering in  $k$ -means depends on the initial seeds. For seeds B and E,  $k$ -means converges to  $\{A, B, C\}, \{D, E, F\}$ , a suboptimal clustering. For seeds D and F, it converges to  $\{A, B, D, E\}, \{C, F\}$ , the global optimum for  $K = 2$ .

# The $K$ -means Algorithm (9/10)

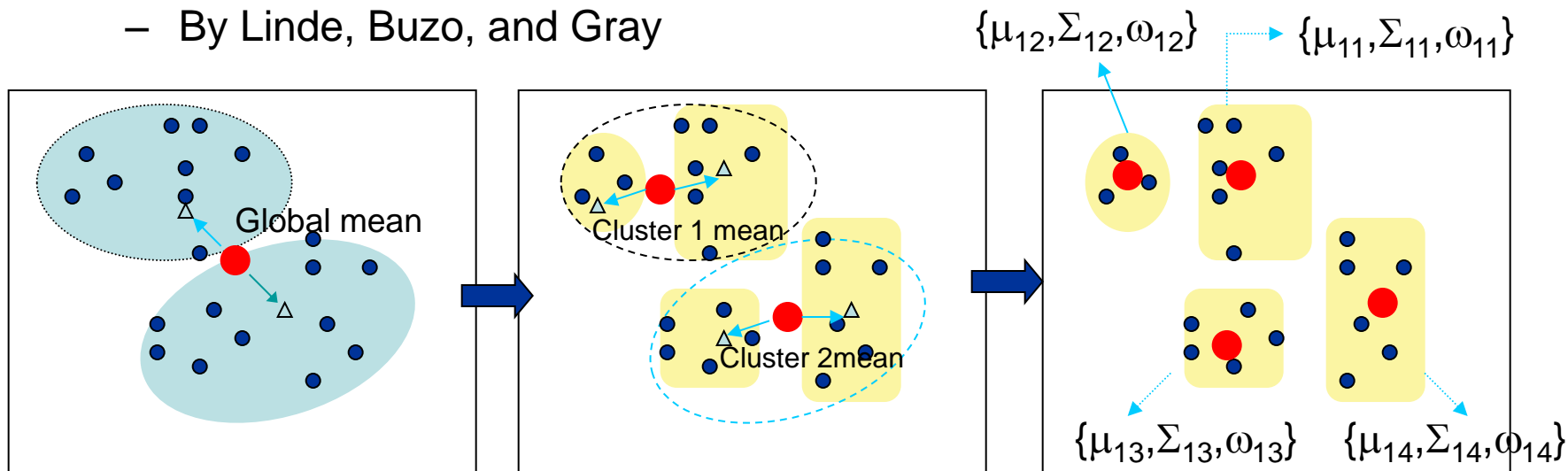
- How to break ties when in case there are several centers with the same distance from an object
  - E.g., randomly assign the object to one of the candidate clusters (or assign the object to the cluster with lowest index)
  - Or, perturb objects slightly
- Possible Applications of the  $K$ -means Algorithm
  - Clustering
  - Vector quantization
  - A preprocessing stage before classification or regression
    - Map from the original space to  $l$ -dimensional space/hypercube

$$l = \log_2 k \quad (k \text{ clusters})$$



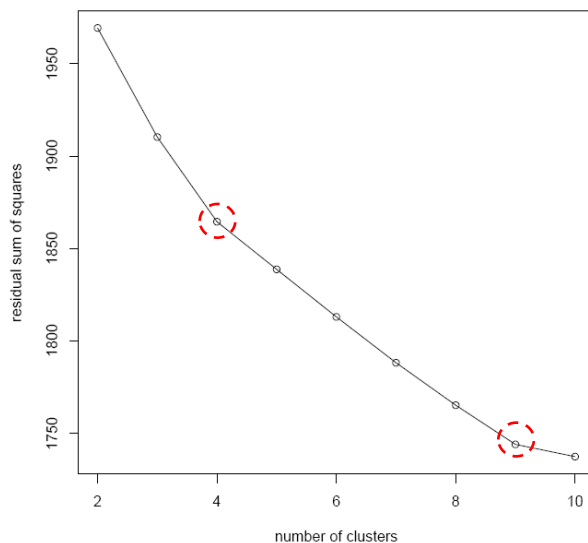
# The $K$ -means Algorithm (10/10)

- E.g., the LBG algorithm  $M \rightarrow 2M$  at each iteration
  - By Linde, Buzo, and Gray



Total reconstruction error  
(residual sum of squares)

$$E(\{\mathbf{m}_i\}_{i=1}^k | \mathbf{X}) = \sum_{t=1}^N \sum_{i=1}^k b_i^t \|\mathbf{x}^t - \mathbf{m}_i\|^2$$

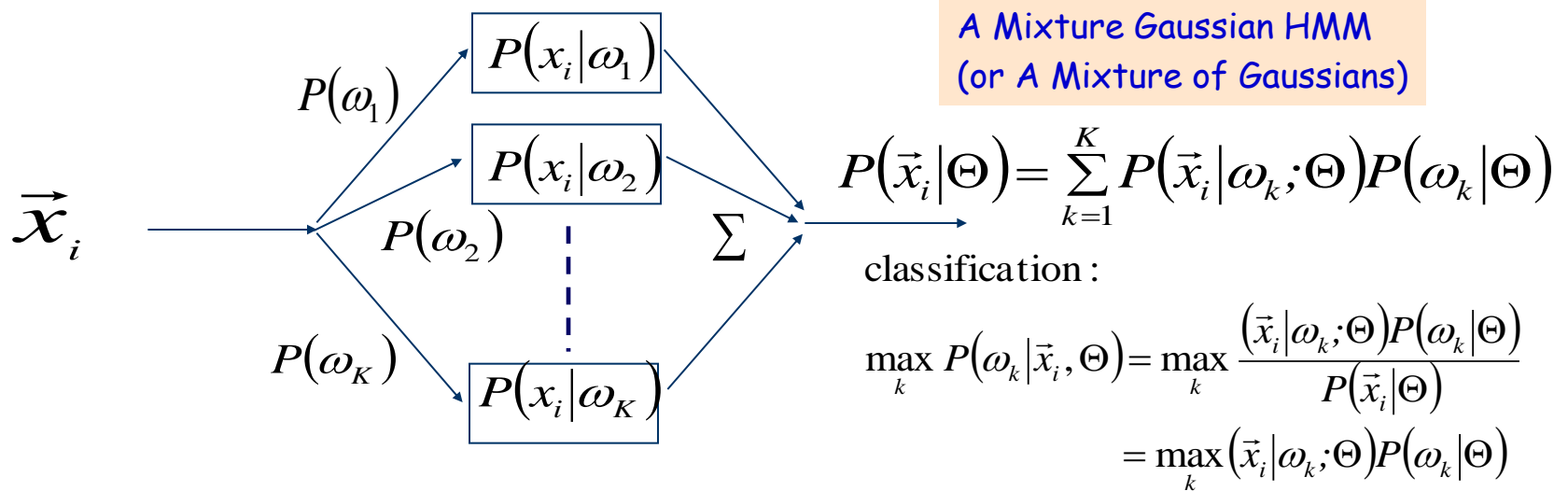


# The EM Algorithm (1/3)

- EM (Expectation-Maximization) algorithm
  - A kind of model-based clustering
  - Also can be viewed as a generalization of  $K$ -means
  - Each cluster is a “model” for generating the data
    - The centroid is good representative for each model
    - Generate an object (e.g., document) consists of first picking a centroid at random and then adding some noise
      - If the noise is normally distributed, the procedure will result in clusters of spherical shape
- Physical Models for EM
  - Discrete: Mixture of multinomial distributions
  - Continuous: Mixture of Gaussian distributions

# The EM Algorithm (2/3)

- EM is a **soft version** of  $K$ -mean
  - Each object could be the member of multiple clusters  $\omega_k$
  - Clustering as estimating a mixture of (continuous) probability distributions



Likelihood function for data samples:  $\mathbf{X} = \vec{x}_1, \vec{x}_2, \dots, \vec{x}_n$

$$P(\mathbf{X} | \Theta) = \prod_{i=1}^n P(\vec{x}_i | \Theta)$$

$$= \prod_{i=1}^n \sum_{k=1}^K P(\vec{x}_i | \omega_k; \Theta) P(\omega_k | \Theta)$$

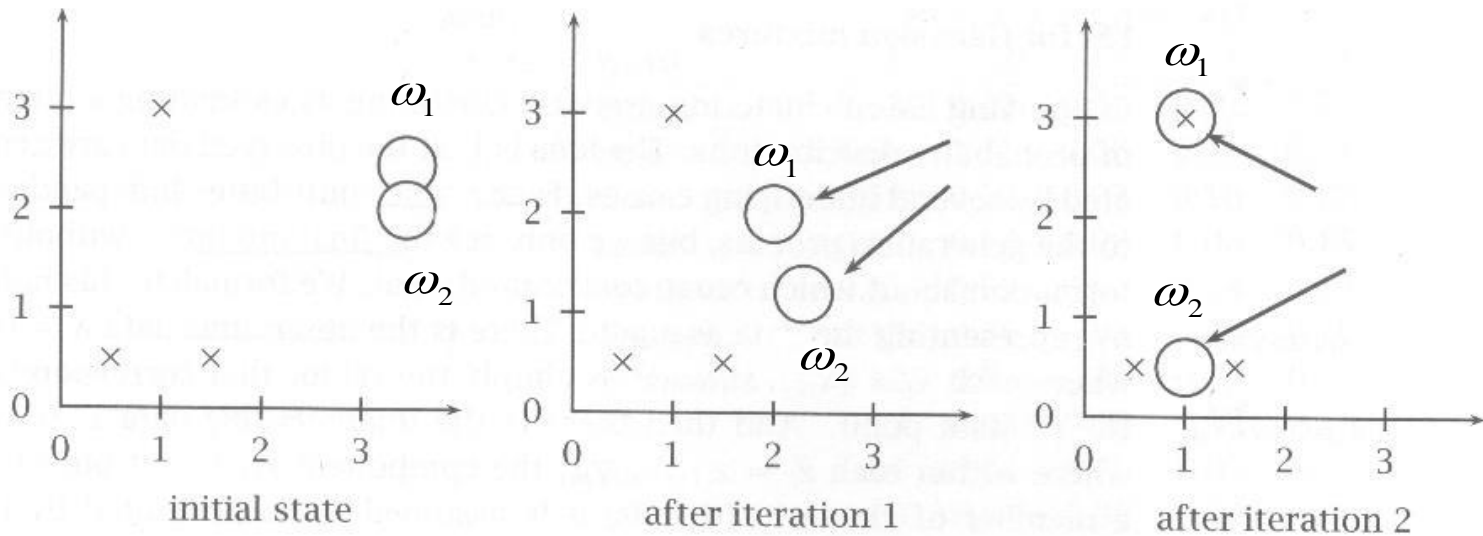
Continuous case:

$$P(\vec{x}_i | \omega_k; \Theta) = \frac{1}{\sqrt{(2\pi)^m |\Sigma_k|}} \exp\left(-\frac{1}{2} (\vec{x}_i - \vec{\mu}_k)^T \Sigma_k^{-1} (\vec{x}_i - \vec{\mu}_k)\right)$$

$$\mathbf{X} = \{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n\}$$

$\vec{x}_i$ 's are independent identically distributed (i.i.d.)

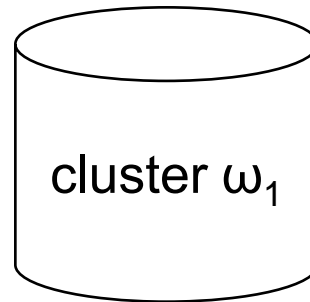
# The EM Algorithm (2/3)



**Figure 14.10** An example of using the EM algorithm for soft clustering.

# Maximum Likelihood Estimation (MLE) (1/2)

- Hard Assignment



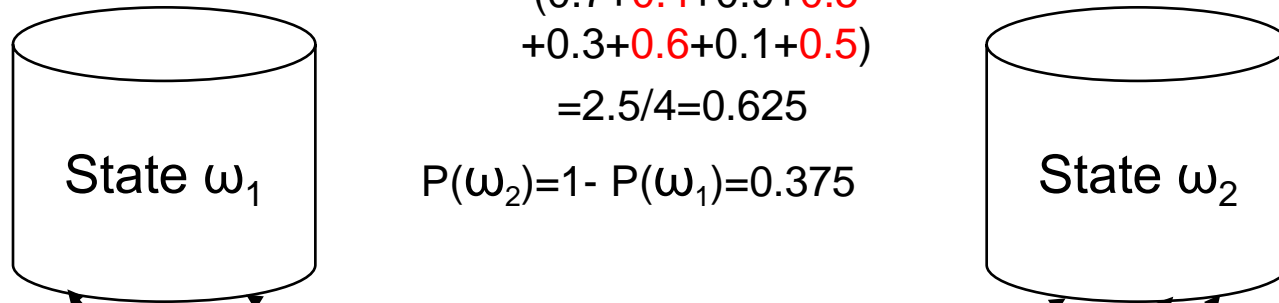
$$P(B | \omega_1) = 2/4 = 0.5$$

$$P(W | \omega_1) = 2/4 = 0.5$$



# Maximum Likelihood Estimation (2/2)

- Soft Assignment



$$P(\omega_1) = \frac{(0.7 + 0.4 + 0.9 + 0.5) + (0.3 + 0.6 + 0.1 + 0.5)}{4} = \frac{2.5}{4} = 0.625$$

$$P(\omega_2) = 1 - P(\omega_1) = 0.375$$

$$P(B|\omega_1) = \frac{(0.7 + 0.9)}{(0.7 + 0.4 + 0.9 + 0.5)} = \frac{1.6}{2.5} = 0.64$$

$$P(B|\omega_1) = \frac{(0.4 + 0.5)}{(0.7 + 0.4 + 0.9 + 0.5)} = \frac{0.9}{2.5} = 0.36$$

$$P(B|\omega_2) = \frac{(0.3 + 0.1)}{(0.3 + 0.6 + 0.1 + 0.5)} = \frac{0.4}{1.5} = 0.27$$

$$P(B|\omega_2) = \frac{(0.6 + 0.5)}{(0.3 + 0.6 + 0.1 + 0.5)} = \frac{0.11}{1.5} = 0.73$$



# Expectation-Maximization Updating Formulas (1/3)

- Expectation

$$\gamma_{ik} = \frac{P(\vec{x}_i | \omega_k, \Theta) P(\omega_k | \Theta)}{\sum_{l=1}^K P(\vec{x}_i | \omega_l, \Theta) P(\omega_l | \Theta)}$$

- Compute the likelihood that each cluster  $\omega_k$  generates a document vector  $\vec{x}_i$

# Expectation-Maximization Updating Formulas (2/3)

- Maximization
  - Mixture Weight

$$P(\omega_k | \hat{\Theta}) = \frac{\sum_{i=1}^n \gamma_{ik}}{\sum_{k'=1}^K \sum_{i=1}^n \gamma_{ik'}} = \frac{\sum_{i=1}^n \gamma_{ik}}{n}$$

- Mean of Gaussian

$$\hat{\mu}_k = \frac{\sum_{i=1}^n \gamma_{ik} \cdot \vec{x}_i}{\sum_{i'=1}^n \gamma_{i'k}}$$

# Expectation-Maximization Updating Formulas (3/3)

- Covariance Matrix of Gaussian

$$\begin{aligned}\hat{\Sigma}_k &= \frac{\sum_{i=1}^n \gamma_{ik} \cdot (\bar{x}_i - \hat{\mu}_k)(\bar{x}_i - \hat{\mu}_k)^T}{\sum_{i'=1}^n \gamma_{i'k}} \\ &= \frac{\sum_{i=1}^n \gamma_{ik} \cdot (\bar{x}_i - \hat{\mu}_k)(\bar{x}_i - \hat{\mu}_k)^T}{\sum_{i'=1}^n \gamma_{i'k}}\end{aligned}$$

# More facts about The EM Algorithm

- The initial cluster distributions can be estimated using the  $K$ -means algorithm, which EM can then “soften up”
- The procedure terminates when the likelihood function  $P(\mathbf{X}|\Theta)$  is converged or maximum number of iterations is reached

# **Hierarchical Clustering**

# Hierarchical Clustering

- Can be in either bottom-up or top-down manners
  - **Bottom-up** (*agglomerative*) 凝集的
    - Start with individual objects and try to group the most similar ones
      - E.g., with the minimum distance apart

$$sim(x, y) = \frac{1}{1 + d(x, y)}$$

distance measures will  
be discussed later on

- The procedure terminates when **one cluster containing all objects** has been formed
- **Top-down** (*divisive*) 分裂的
  - Start with all objects in a group and divide them into groups so as to maximize **within-group** similarity

# Hierarchical Agglomerative Clustering (HAC)

- A bottom-up approach
- Assume a similarity measure for determining the similarity of two objects
- Start with all objects in a separate cluster (a singleton) and then repeatedly joins the two clusters that have the most similarity until there is only one cluster survived
- The history of merging/clustering forms a binary tree or hierarchy

# HAC: Algorithm

```
1 Given: a set  $\mathcal{X} = \{x_1, \dots, x_n\}$  of objects
2       a function  $\text{sim}: \mathcal{P}(\mathcal{X}) \times \mathcal{P}(\mathcal{X}) \rightarrow \mathbb{R}$ 
3 for  $i := 1$  to  $n$  do      Initialization (for tree leaves):
4    $c_i := \{x_i\}$  end      Each object is a cluster
5  $C := \{c_1, \dots, c_n\}$ 
6  $j := n + 1$ 
7 while  $|C| > 1$       cluster number
8    $(c_{n_1}, c_{n_2}) := \arg \max_{(c_u, c_v) \in C \times C} \text{sim}(c_u, c_v)$ 
9    $c_j = c_{n_1} \cup c_{n_2}$       merged as a new cluster
10   $C := C \setminus \{c_{n_1}, c_{n_2}\} \cup \{c_j\}$       The original two clusters
11   $j := j + 1$       are removed
```

Figure 14.2 Bottom-up hierarchical clustering.

- $c_i$  denotes a specific cluster here



# Distance Metrics

- Euclidian Distance ( $L_2$  norm)

$$L_2(\vec{x}, \vec{y}) = \sum_{i=1}^m (x_i - y_i)^2$$

- Make sure that all attributes/dimensions have the same scale (or the same variance)

- $L_1$  Norm (City-block distance)

$$L_1(\vec{x}, \vec{y}) = \sum_{i=1}^m |x_i - y_i|$$

- Cosine Similarity (transform to a distance by subtracting from 1)

$$1 - \frac{\vec{x} \cdot \vec{y}}{|\vec{x}| \cdot |\vec{y}|}$$

ranged between 0 and 1

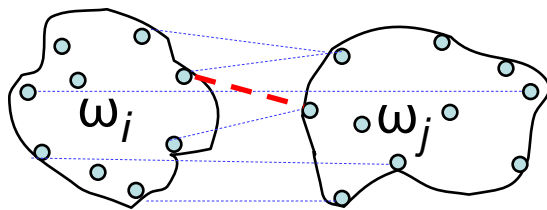
# Measures of Cluster Similarity (1/9)

- Especially for the bottom-up approaches

## 1. Single-link clustering

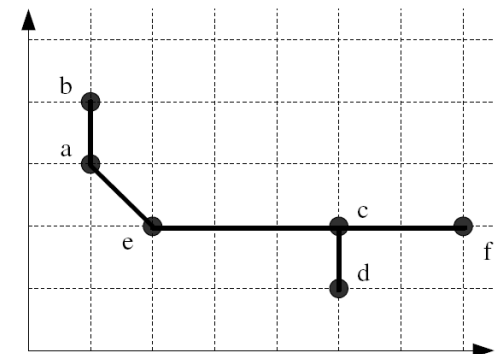
- The similarity between two clusters is the similarity of the two closest objects in the clusters
- Search over all pairs of objects that are from the two different clusters and select the pair with the greatest similarity
- Elongated clusters are achieved

$$\text{sim}(\omega_i, \omega_j) = \max_{\vec{x} \in \omega_i, \vec{y} \in \omega_j} \text{sim}(\vec{x}, \vec{y})$$



greatest similarity

cf. the minimal  
spanning tree

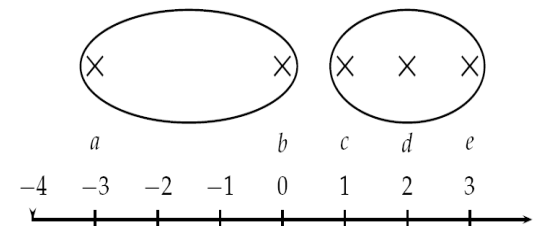
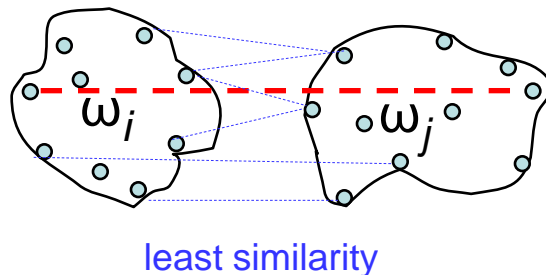


# Measures of Cluster Similarity (2/9)

## 2. Complete-link clustering

- The similarity between two clusters is the similarity of their two most dissimilar members
- Sphere-shaped clusters are achieved
- Preferable for most IR and NLP applications

$$\text{sim}(\omega_i, \omega_j) = \min_{\vec{x} \in \omega_i, \vec{y} \in \omega_j} \text{sim}(\vec{x}, \vec{y})$$



- More sensitive to outliers

► **Figure 17.6** Outliers in complete-link clustering. The four points have the coordinates  $-3 + 2 \times \epsilon, 0, 1 + 2 \times \epsilon, 2$  and  $3 - \epsilon$ . Complete-link clustering creates the two clusters shown as ellipses. Intuitively,  $\{b, c, d, e\}$  should be one cluster, but it is split by outlier  $a$ .

# Measures of Cluster Similarity (3/9)

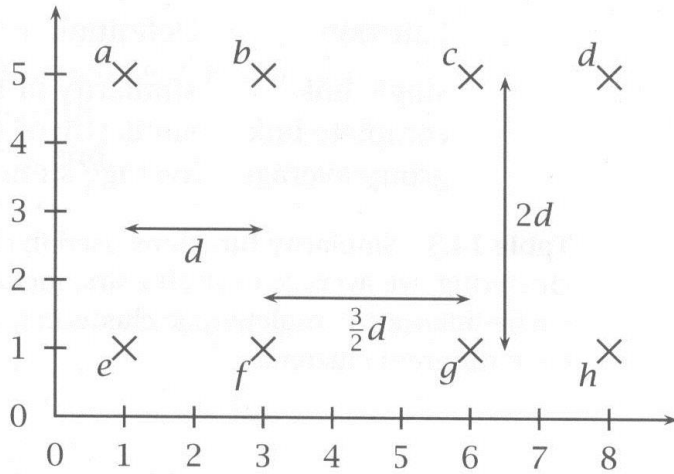


Figure 14.4 A cloud of points in a plane.

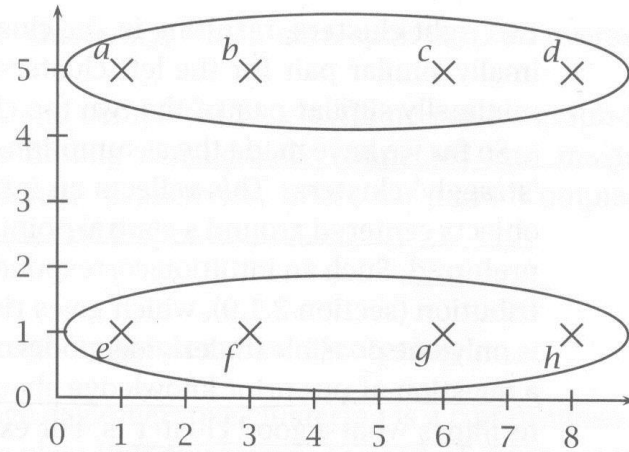


Figure 14.6 Single-link clustering of the points in figure 14.4.

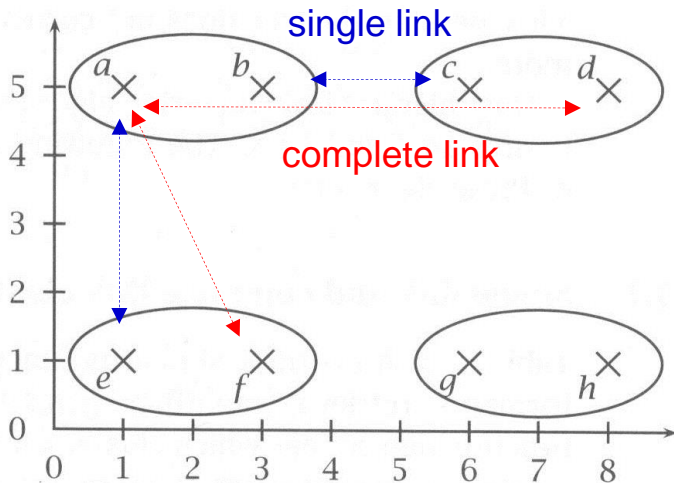


Figure 14.5 Intermediate clustering of the points in figure 14.4.

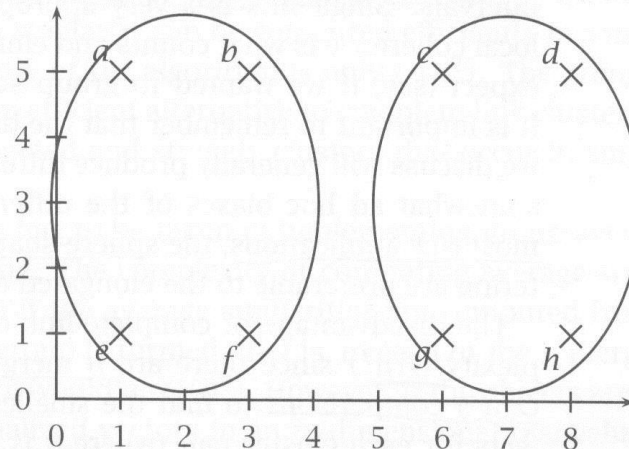
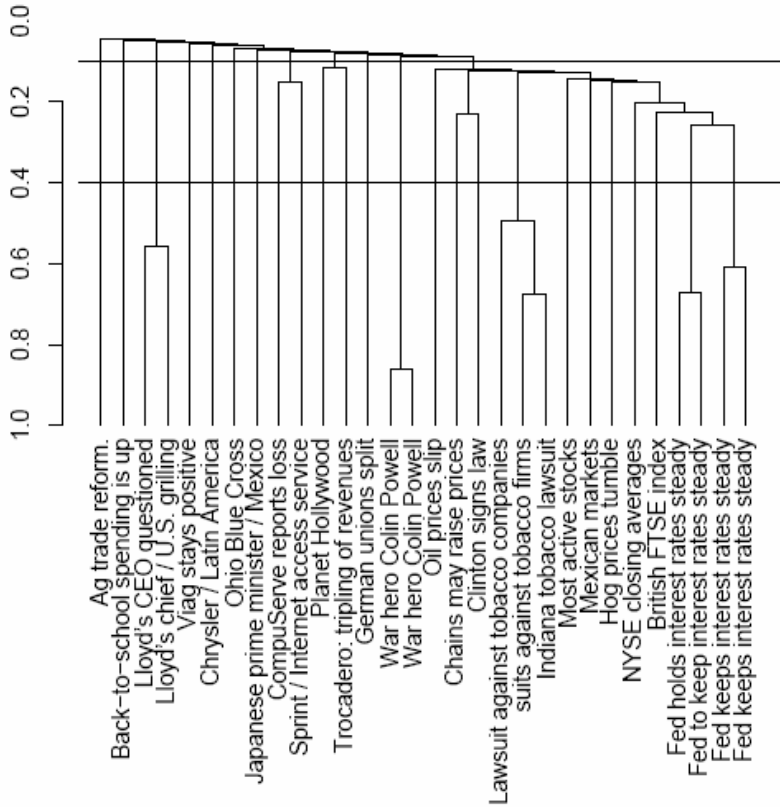
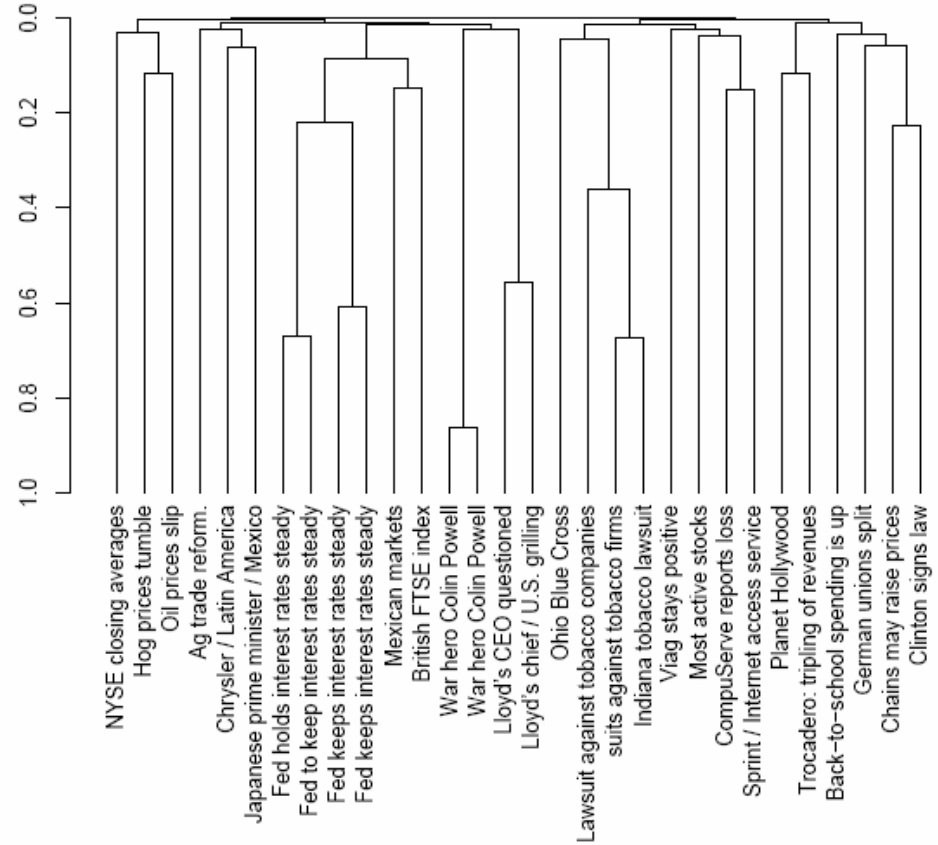


Figure 14.7 Complete-link clustering of the points in figure 14.4.

# Measures of Cluster Similarity (4/9)



► **Figure 17.1** A dendrogram of a single-link clustering of 30 documents from Reuters-RCV1. The y-axis represents combination similarity, the similarity of the two component clusters that gave rise to the corresponding merge. For example, the combination similarity of *Lloyd's CEO questioned* and *Lloyd's chief / U.S. grilling* is  $\approx 0.56$ . Two possible cuts of the dendrogram are shown: at 0.4 into 24 clusters and at 0.1 into 12 clusters.

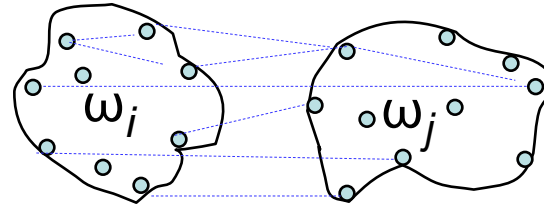


► **Figure 17.4** A dendrogram of a complete-link clustering of 30 documents from Reuters-RCV1. This complete-link clustering is more balanced than the single-link clustering of the same documents in Figure 17.1. When cutting the last merger, we obtain two clusters of similar size (documents 1–16 and documents 17–30). The y-axis represents combination similarity.

# Measures of Cluster Similarity (5/9)

## 3. **Group-average** agglomerative clustering

- A compromise between single-link and complete-link clustering
- The similarity between two clusters is the average similarity between members



- If the objects are represented as length-normalized vectors and the similarity measure is the cosine
  - There exists an fast algorithm for computing the average similarity

$$\text{sim}(\vec{x}, \vec{y}) = \cos(\vec{x}, \vec{y}) = \frac{\vec{x} \cdot \vec{y}}{|\vec{x}| |\vec{y}|} = \vec{x} \cdot \vec{y}$$

length-normalized vectors

# Measures of Cluster Similarity (6/9)

## 3. Group-average agglomerative clustering (cont.)

- The average similarity *SIM* between vectors in a cluster  $\omega_j$  is defined as

$$SIM(\omega_j) = \frac{1}{|\omega_j|(|\omega_j| - 1)} \sum_{\substack{\vec{x} \in \omega_j \\ \vec{y} \in \omega_j \\ \vec{y} \neq \vec{x}}} sim(\vec{x}, \vec{y}) = \frac{1}{|\omega_j|(|\omega_j| - 1)} \sum_{\vec{x} \in \omega_j} \sum_{\substack{\vec{y} \in \omega_j \\ \vec{y} \neq \vec{x}}} \vec{x} \cdot \vec{y}$$

- The sum of members in a cluster  $\omega_j$ :  $\vec{s}(\omega_j) = \sum_{\vec{x} \in \omega_j} \vec{x}$

- Express  $SIM(\omega_j)$  in terms of  $\vec{s}(\omega_j)$

$$\vec{s}(\omega_j) \cdot \vec{s}(\omega_j) = \sum_{\vec{x} \in \omega_j} \vec{x} \cdot \vec{s}(\omega_j) = \sum_{\vec{x} \in \omega_j} \sum_{\vec{y} \in \omega_j} \vec{x} \cdot \vec{y}$$

length-normalized vector

$$= |\omega_j|(|\omega_j| - 1)SIM(\omega_j) + \sum_{\vec{x} \in \omega_j} \vec{x} \cdot \vec{x} = 1$$

$$= |\omega_j|(|\omega_j| - 1)SIM(\omega_j) + |\omega_j|$$

$$\therefore SIM(c_j) = \frac{\vec{s}(\omega_j) \cdot \vec{s}(\omega_j) - |\omega_j|}{|\omega_j|(|\omega_j| - 1)}$$

# Measures of Cluster Similarity (7/9)

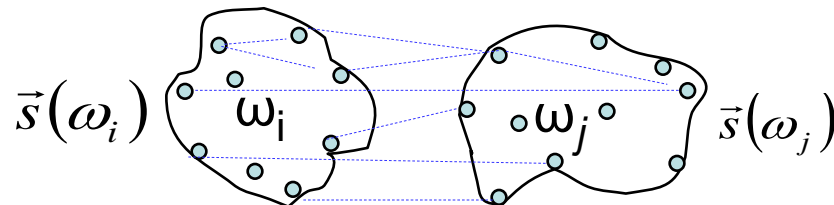
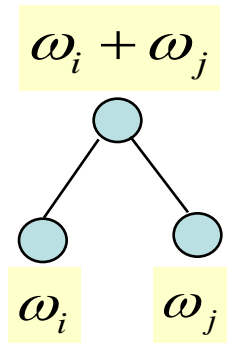
## 3. Group-average agglomerative clustering (cont.)

-As merging two clusters  $c_i$  and  $c_j$ , the cluster sum vectors  $\vec{s}(\omega_i)$  and  $\vec{s}(\omega_j)$  are known in advance

$$\Rightarrow \vec{s}(\omega_{New}) = \vec{s}(\omega_i) + \vec{s}(\omega_j), \quad |\omega_{New}| = |\omega_i| + |\omega_j|$$

- The average similarity for their union will be

$$SIM(\omega_i \cup \omega_j) = \frac{(\vec{s}(\omega_i) + \vec{s}(\omega_j)) \cdot (\vec{s}(\omega_i) + \vec{s}(\omega_j)) - (|\omega_i| + |\omega_j|)}{(|\omega_i| + |\omega_j|)(|\omega_i| + |\omega_j| - 1)}$$





# Measures of Cluster Similarity (8/9)

## 4. Centroid clustering

- The similarity of two clusters is defined as the similarity of their centroids

$$\begin{aligned} \text{sim}(\omega_i, \omega_j) &= \vec{\mu}(\omega_i) \cdot \vec{\mu}(\omega_j) \\ &= \left( \frac{1}{N_i} \sum_{\vec{x}_s \in \omega_i} \vec{x}_s \right) \cdot \left( \frac{1}{N_j} \sum_{\vec{x}_t \in \omega_j} \vec{x}_t \right) \\ &= \frac{1}{N_i N_j} \sum_{\vec{x}_s \in \omega_i} \sum_{\vec{x}_t \in \omega_j} \vec{x}_s \cdot \vec{x}_t \end{aligned}$$

# Measures of Cluster Similarity (9/9)

- Graphical summary of four cluster similarity measures

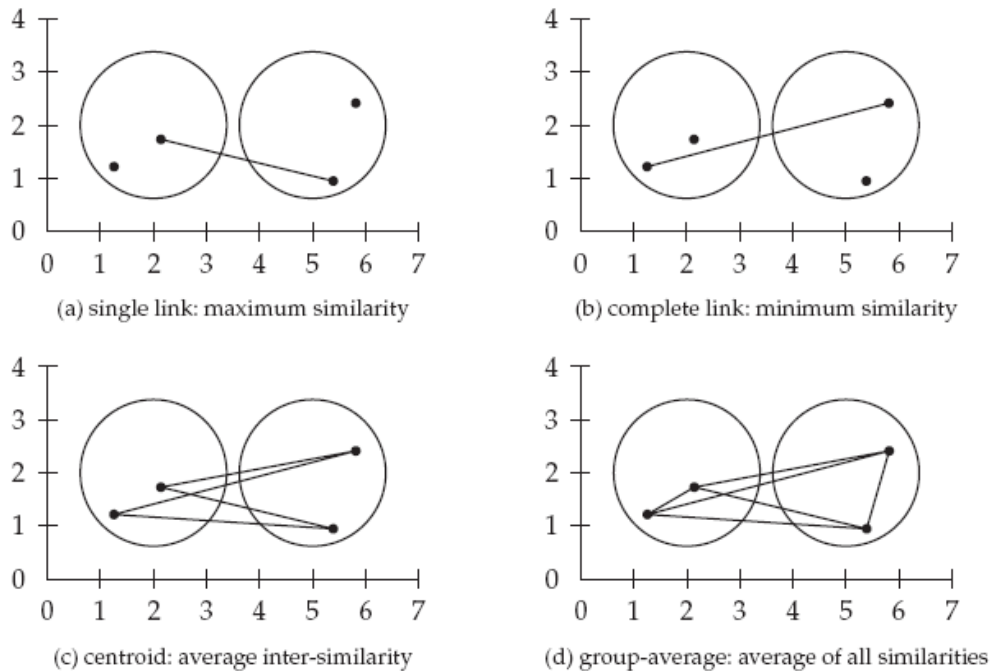
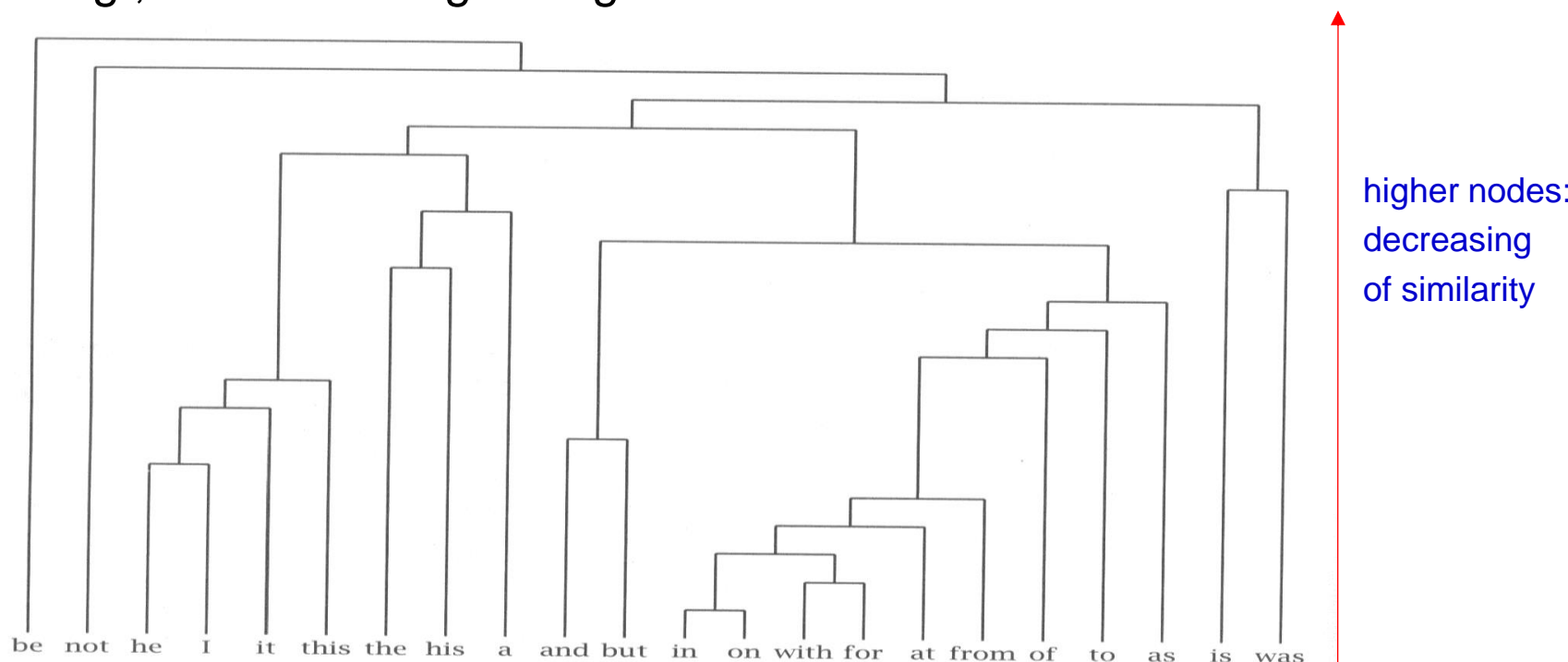


Figure 17.3 The different notions of cluster similarity used by the four HAC algorithms. An *inter-similarity* is a similarity between two documents from different clusters.

clustering algorithm	$SIM(i, k_1, k_2)$
single-link	$\max(SIM(i, k_1), SIM(i, k_2))$
complete-link	$\min(SIM(i, k_1), SIM(i, k_2))$
centroid	$(\frac{1}{N_m} \vec{v}_m) \cdot (\frac{1}{N_i} \vec{v}_i)$
group-average	$\frac{1}{(N_m + N_i)(N_m + N_i - 1)} [(\vec{v}_m + \vec{v}_i)^2 - (N_m + N_i)]$

# Example: Word Clustering

- Words (objects) are described and clustered using a set of features and values
  - E.g., the left and right neighbors of tokens of words



**Figure 14.1** A single-link clustering of 22 frequent English words represented as a dendrogram.

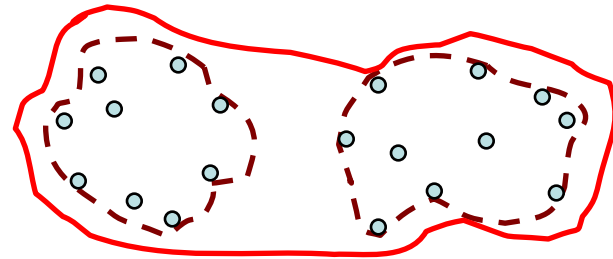
"be" has least similarity with the other 21 words !

# Divisive Clustering (1/2)

- A top-down approach
- Start with all objects in a single cluster
- At each iteration, select the least **coherent** cluster and **split** it
- Continue the iterations until a predefined criterion (e.g., the cluster number) is achieved
- The history of clustering forms a binary tree or hierarchy

## Divisive Clustering (2/2)

- To select the least **coherent** cluster, the measures used in bottom-up clustering (e.g. HAC) can be used again here
  - Single link measure
  - Complete-link measure
  - Group-average measure



- How to **split** a cluster
  - **Also is a clustering task** (finding two sub-clusters)
  - Any clustering algorithm can be used for the splitting operation, e.g.,
    - Bottom-up (agglomerative) algorithms
    - Non-hierarchical clustering algorithms (e.g., *K*-means)

# Divisive Clustering: Algorithm

```
1 Given: a set  $\mathcal{X} = \{x_1, \dots, x_n\}$  of objects
2       a function  $\text{coh}: \mathcal{P}(\mathcal{X}) \rightarrow \mathbb{R}$ 
3       a function  $\text{split}: \mathcal{P}(\mathcal{X}) \rightarrow \mathcal{P}(\mathcal{X}) \times \mathcal{P}(\mathcal{X})$ 
4  $C := \{\mathcal{X}\}$  ( $= \{c_1\}$ )
5  $j := 1$ 
6 while  $\exists c_i \in C$  s.t.  $|c_i| > 1$ 
7      $c_u := \arg \min_{c_v \in C} \text{coh}(c_v)$ 
8      $(c_{j+1}, c_{j+2}) := \text{split}(c_u)$ 
9      $C := C \setminus \{c_u\} \cup \{c_{j+1}, c_{j+2}\}$ 
10     $j := j + 2$ 
```

split the least coherent cluster

Generate two new clusters and remove the original one

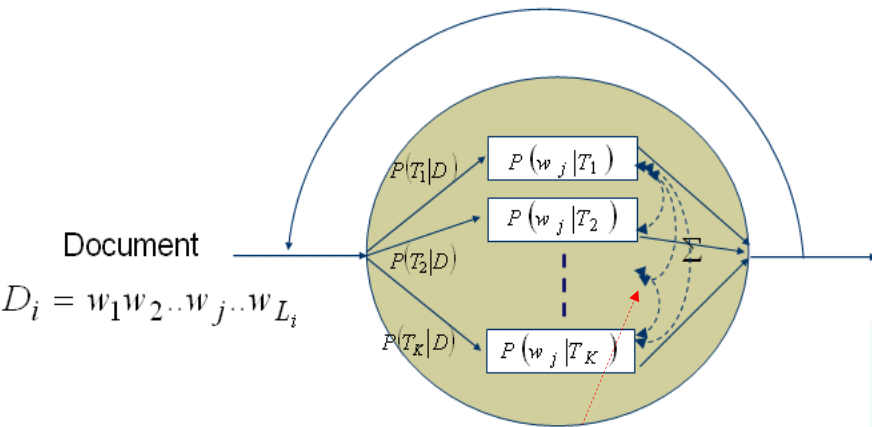
Figure 14.3 Top-down hierarchical clustering.

- $c_u$  denotes a specific cluster here

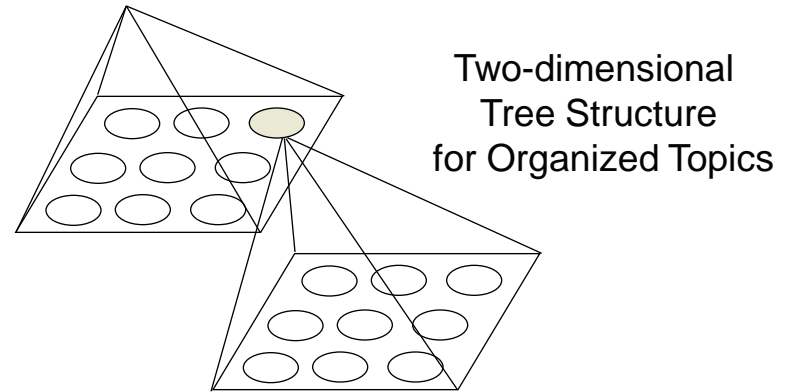
# Hierarchical Document Organization (1/7)

- Explore the Probabilistic Latent Topical Information
  - TMM/PLSA approach

A document model



$$P(w_j | D_i) = \sum_{k=1}^K P(T_k | D_i) \left[ \sum_{l=1}^K P(T_l | Y_k) P(w_j | T_l) \right]$$



$$dist(T_i, T_j) = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2} \quad E(T_l, T_k) = \frac{1}{\sqrt{2\pi\sigma}} \exp\left[-\frac{dist(T_k, T_l)^2}{2\sigma^2}\right]$$

$$P(T_l | Y_k) = \frac{E(T_l, T_k)}{\sum_{s=1}^K E(T_s, T_k)}$$

- Documents are clustered by the latent topics and organized in a two-dimensional tree structure, or a two-layer map
- Those related documents are in the same cluster and the relationships among the clusters have to do with the distance on the map
- When a cluster has many documents, we can further analyze it into an other map on the next layer

# Hierarchical Document Organization (2/7)

- The model can be trained by maximizing the total log-likelihood of all terms observed in the document collection

$$\begin{aligned}
 L_T &= \sum_{i=1}^N \sum_{n=1}^J c(w_j, D_i) \log P(w_j | D_i) \\
 &= \sum_{i=1}^N \sum_{n=1}^J c(w_j, D_i) \log \left\{ \sum_{k=1}^K P(T_k | D_i) \left[ \sum_{l=1}^K P(T_l | Y_k) P(w_j | T_l) \right] \right\}
 \end{aligned}$$

- EM training can be performed

$$\hat{P}(w_j | T_k) = \frac{\sum_{i=1}^N c(w_j, D_i) P(T_k | w_j, D_i)}{\sum_{j'=1}^J \sum_{i'=1}^N c(w_{j'}, D_{i'}) P(T_k | w_{j'}, D_{i'})}$$

$$\hat{P}(T_k | D_i) = \frac{\sum_{j=1}^J c(w_j, D_i) P(T_k | w_j, D_i)}{c(D_i)}$$

where

$$P(T_k | w_j, D_i) = \frac{\left[ \sum_{l=1}^K P(w_j | T_l) P(T_l | T_k) \right] \cdot P(T_k | D_i)}{\sum_{k'=1}^K \left\{ \left[ \sum_{l'=1}^K P(w_j | T_{l'}) P(T_{l'} | T_{k'}) \right] \cdot P(T_{k'} | D_i) \right\}}$$



# Hierarchical Document Organization (3/7)

- Criterion for Topic Word Selecting
  - Topic Ratio

$$S(w_j, T_k) = \frac{\sum_{i=1}^N c(w_j, D_i) P(T_k | D_i)}{\sum_{i'=1}^N c(w_j, D_{i'}) [1 - P(T_k | D_{i'})]}$$

# Hierarchical Document Organization (4/7)

- Example

The screenshot displays a web browser window with the following content:

**Left Sidebar: Title/Summary Generation Demo System**

- National Taiwan University
- Speech Processing Laboratory
- 國外政治 [Topic Map](#)
- 國內國會 [Topic Map](#)
- 國外社會 [Topic Map](#)
- 國外財經 [Topic Map](#)
- 國內財經 [Topic Map](#)
- 地方政府 [Topic Map](#)
- 國內政治 [Topic Map](#)
- 國內交通 [Topic Map](#)
- 國內影劇 [Topic Map](#)
- 國外體育 [Topic Map](#)
- 國內社會 [Topic Map](#)
- 大陸社會 [Topic Map](#)
- 國外醫藥 [Topic Map](#)
- 國外影劇 [Topic Map](#)
- 大陸財經 [Topic Map](#)
- 國內文教 [Topic Map](#)
- 國內體育 [Topic Map](#)
- 國內醫藥 [Topic Map](#)
- 大陸政治 [Topic Map](#)

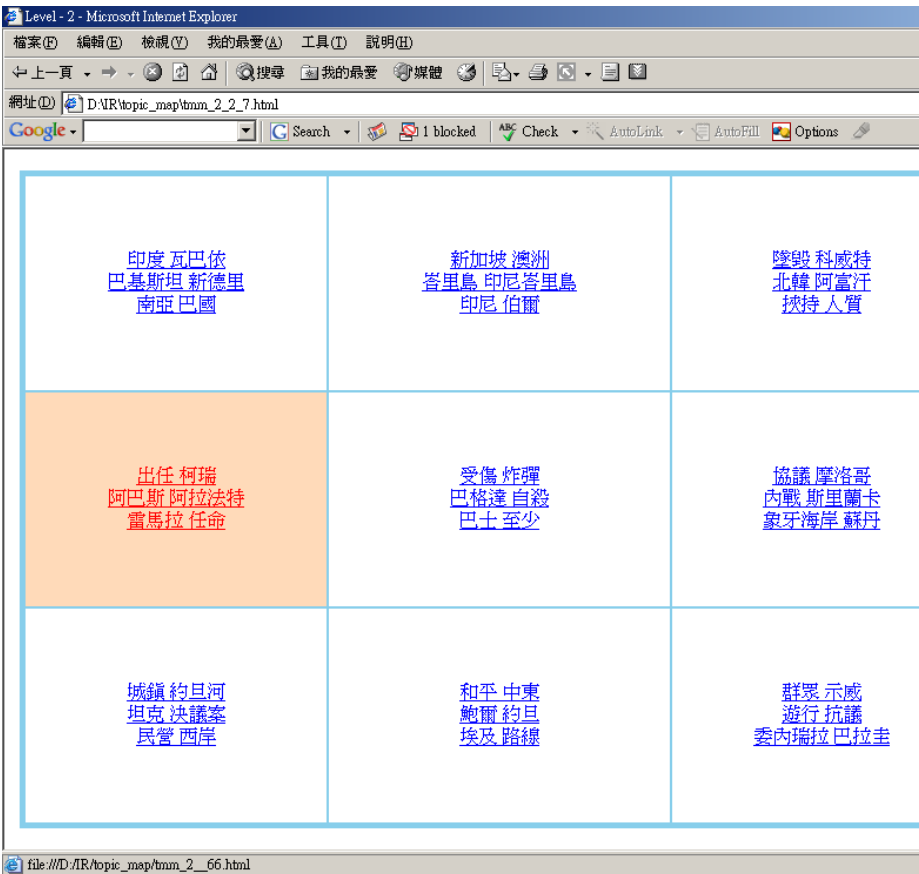
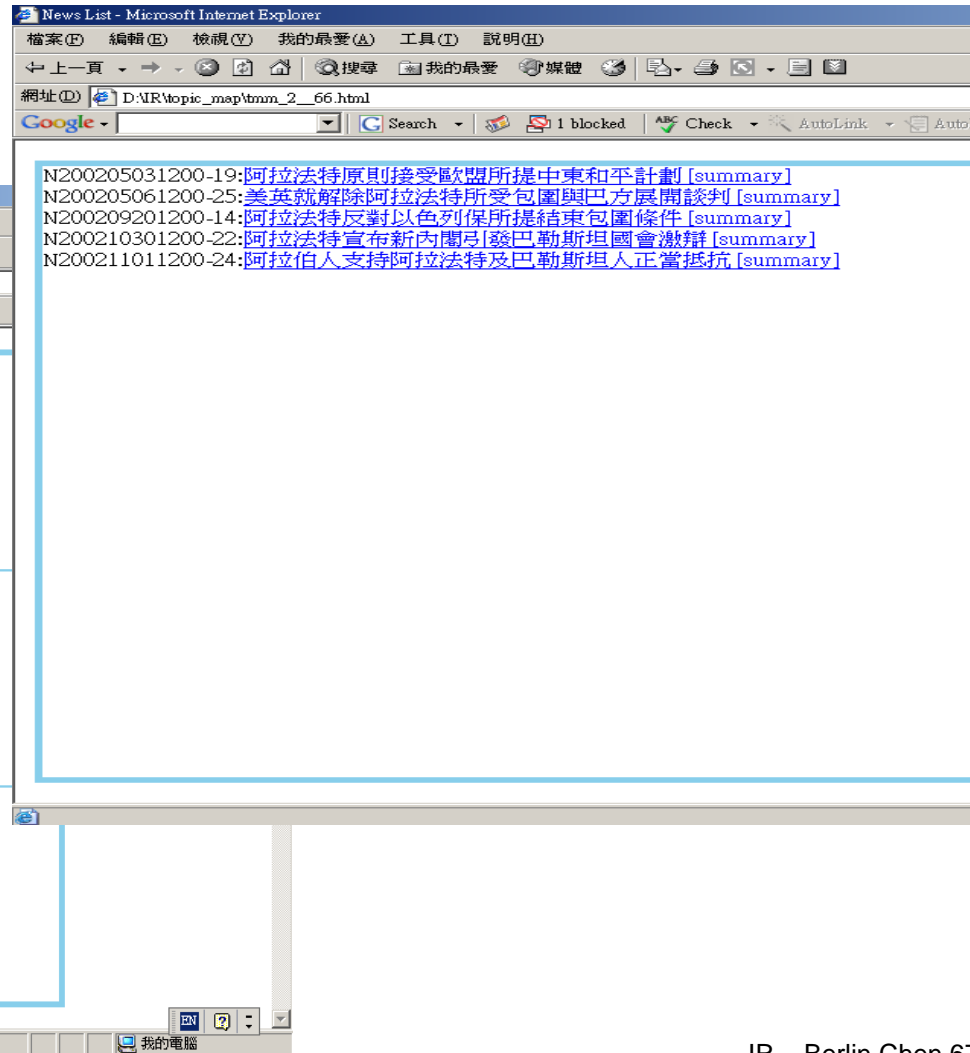
**Main Content Area: Hierarchical Document Organization (3x3 Grid)**

<p>聯邦調查局 執法 劃歸 空對空 飛彈 安全部 艾希克羅 蓋達組織 接種 等級 民航機 認出 輻射性 劫機 主謀 重整旗鼓 歐瑪 穆勒 國土 黃色 葉門 美國境內 申請局 天花 芮吉</p>	<p>僑界 僑務 台商 會長 僑胞 呼吸 雙十國慶 酒會 立委 舉辦 國慶 聯誼會 經文 履新 組長 衛生 餐會 春館 滬太華 後援 中華 僑團 華僑 鄉親</p>	<p>法輪 鈴木宗男 巫統 中國共產黨 李光耀 挪用 書記 交替 班子 馬哈地 一邊 李顯龍 吳作棟 新疆 論說 軍委 政治局 標題 馬來人 早報 格局 資政 接班 報章</p>
<p>檢查人員 檢查員 動武 最後通牒 安理會 布里克斯 決議 精密 武檢 聯合國 授權 沙丹 銷毀 違禁 解除 武檢人員 檢查 首席 武器 決議案 胡筭 禁航區 導引 毀滅性</p>	<p>西非 衛隊 巴格達機場 伊拉克部隊 伊拉克南部 賴比瑞亞 伊北 科威特 步兵 辛格 庫德族 斯拉 法新社 翁山蘇姬 庫克 蒙羅維亞 巴格達 陸戰隊 轟炸 激戰 卡達 克里 市中心 基爾</p>	<p>林東源 金大中 漢城 南北 多邊 正常化 長官 平壤 分界線 會談 鐵路 南韓 統一 韓美 燃料 南韓 懸案 金正日 盧武鉉 朝鮮 半島 打撈 黃海 銜接 核子 北韓</p>
<p>普查 支領 王太 王室 登基 會計年度 小泉內閣 瑪格麗特 問卷 靈樞 溫莎堡 英鎊 西敏寺 大廳 白金漢宮 社會勞工黨 王太后 加班 女王 降至 百分點 享年 伊麗莎白 太后 大關</p>	<p>自殺 加薩市 炸彈 巴勒斯坦 賊鎮 約旦河 巴勒斯坦人 哈瑪斯 喪生 耶路撒冷 阿拉法特 約旦河西岸 以色列 伯利恆 槍手 加薩走廊 夏隆 黎區 西岸 受傷 特拉維夫 以色列部隊 包圍 巴士</p>	<p>中美洲 決選 薩爾瓦多 哥斯大黎加 中間 兼職 雷朋 宏都拉斯 羅育 馬達加斯加 史瓦濟蘭 翁岳生 王金平 勳章 院長 金哥納 馬拉坎南宮 游錫方 右派 雅羅 查維斯 喬斯班 孟代爾 方士</p>

**Bottom Status Bar:** file:///D:/R/topic\_map/tmm\_2\_2\_7.html

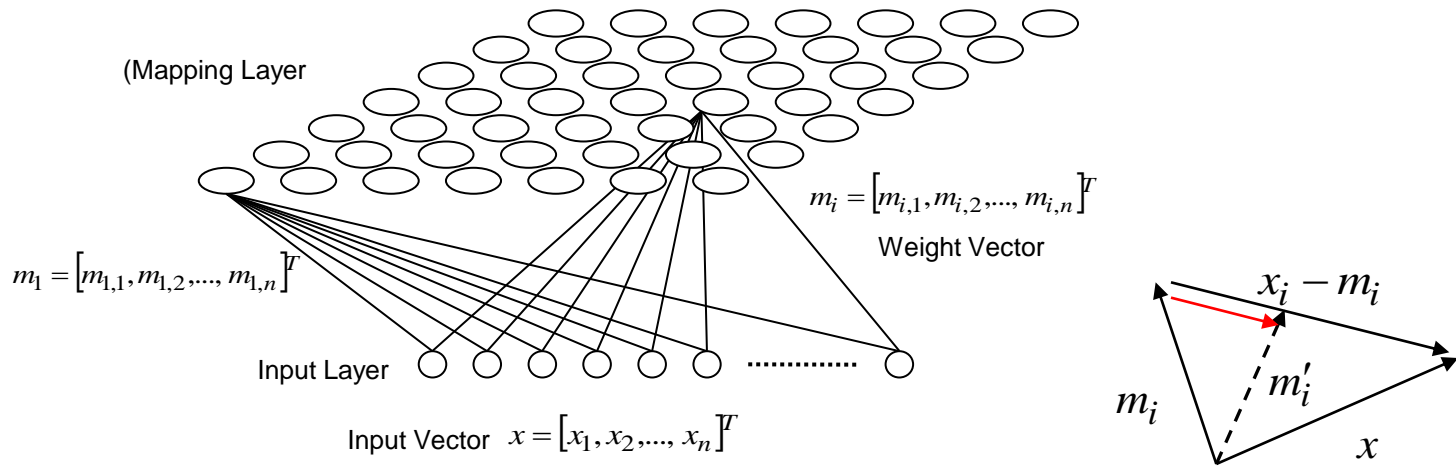
# Hierarchical Document Organization (5/7)

- Example (cont.)



# Hierarchical Document Organization (6/7)

- Self-Organization Map (SOM)
  - A recursive regression process



$$m_i(t+1) = m_i(t) + h_{c(x),i}(t)[x(t) - m_i(t)]$$

$$c(x) = \arg \min_{i'} \|x - m_{i'}\|$$

where

$$\|x - m_{i'}\| = \sqrt{\sum_n (x_n - m_{i',n})^2}$$

$$h_{c(x),i}(t) = \alpha(t) \exp\left(-\frac{\|r_i - r_{c(x)}\|^2}{2\sigma^2(t)}\right)$$

# Hierarchical Document Organization (7/7)

- Results

Model	Iterations	$dist_{Between}/dist_{Within}$
TMM	10	1.9165
	20	2.0650
	30	1.9477
	40	1.9175
SOM	100	2.0604

$$R_{Dist} = \frac{dist_{Between}}{dist_{Within}}$$

where

$$dist_{Between} = \frac{\sum_{i=1}^{|D|} \sum_{j=i+1}^{|D|} f_{Between}(i, j)}{\sum_{i=1}^{|D|} \sum_{j=i+1}^{|D|} C_{Between}(i, j)}$$

$$dist_{Within} = \frac{\sum_{i=1}^{|D|} \sum_{j=i+1}^{|D|} f_{Within}(i, j)}{\sum_{i=1}^{|D|} \sum_{j=i+1}^{|D|} C_{Within}(i, j)}$$

$$f_{Between}(i, j) = \begin{cases} dist_{Map}(i, j) & T_{r,i} \neq T_{r,j} \\ 0 & otherwise \end{cases}$$

$$dist_{Map}(i, j) = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}$$

$$C_{Between}(i, j) = \begin{cases} 1 & T_{r,i} \neq T_{r,j} \\ 0 & otherwise \end{cases}$$

$$f_{Within}(i, j) = \begin{cases} dist_{Map}(i, j) & T_{r,i} = T_{r,j} \\ 0 & otherwise \end{cases}$$

$$C_{Within}(i, j) = \begin{cases} 1 & T_{r,i} = T_{r,j} \\ 0 & otherwise \end{cases}$$