Discrete Random Variables: Expectation, Mean and Variance

Berlin Chen
Department of Computer Science & Information Engineering
National Taiwan Normal University

Reference:

- D. P. Bertsekas, J. N. Tsitsiklis, *Introduction to Probability* , Sections 2.3-2.4

Motivation (1/2)

- **An Illustrative Example**: Suppose that you spin the wheel k times, and that k_i is the number of times that the outcome (money) is m_i (there are n distinct outcomes, $m_1, m_2, \ldots, m_n$)

- What is the amount of money that you "*expect*" to get "*per spin*"?
  - The total amount received is

$$m_1 k_1 + m_2 k_2 + \cdots + m_n k_n$$

  - The amount received per spin is

$$M = \frac{m_1 k_1 + m_2 k_2 + \cdots + m_n k_n}{k}$$

Motivation (2/2)

- If the number of spins k is very large, and if we are willing to interpret probabilities as relative frequencies, it is reasonable to anticipate that m_i comes up a fraction of times that is roughly equal to p_i

$$p_i \approx \frac{k_i}{k}$$

- Therefore, the amount received per spin can be also represented as

$$M = \frac{m_1 k_1 + m_2 k_2 + \cdots + m_n k_n}{k}$$

$$= m_1 p_1 + m_2 p_2 + \cdots + m_n p_n$$

Expectation

- The **expected value** (also called the **expectation** or the **mean**) of a random variable X, with PMF p_X, is defined by

$$\mathbf{E}[X] = \sum_x x p_X(x)$$

  – Can be interpreted as the **center of gravity** of the PMF

- The expectation is well-defined if

$$\sum_x |x| p_X(x) < \infty$$

  – That is, $\sum_x x p_X(x)$ converges to a finite value

Moments

- The ***n*-th moment** of a random variable X is the expected value of a random variable X^n (or the random variable Y , $Y = g(X) = X^n$)

$$\mathbf{E}\left[X^n\right] = \sum_x x^n p_X(x)$$

  - The 1st moment of a random variable X is just its mean (or expectation)

Expectations for Functions of Random Variables

- Let X be a random variable with PMF p_X, and let $g(X)$ be a function of X. Then, the expected value of the random variable $g(X)$ is given by

$$\mathbf{E}[g(X)] = \sum_x g(x) p_X(x)$$

- To verify the above rule
  - Let $Y = g(X)$, and therefore $p_Y(y) = \sum_{\{x \mid g(x) = y\}} p_X(x)$

$$\mathbf{E}[g(X)] = \mathbf{E}[Y] = \sum_y y p_Y(y)$$

$$= \sum_y y \sum_{\{x \mid g(x) = y\}} p_X(x) = \sum_y \sum_{\{x \mid g(x) = y\}} g(x) p_X(x)$$

$$= \sum_x g(x) p_X(x)$$

Variance

- The **variance** of a random variable X is the expected value of a random variable $(X - \mathbf{E}(X))^2$

$$\operatorname{var}(X) = \mathbf{E}\left[(X - \mathbf{E}[X])^2\right]$$
$$= \sum_x (X - \mathbf{E}[X])^2 p_X(x)$$

  – The variance is always nonnegative (why?)
  – The variance provides a measure of dispersion of X around its mean
  – The standard derivation is another measure of dispersion, which is defined as (a square root of variance)

$$\sigma_X = \sqrt{\operatorname{var}(X)}$$

    - Easier to interpret, because it has the same units as X

An Example

- **Example 2.3:** For the random variable X with PMF

$$p_X(x) = \begin{cases} 1/9, & \text{if x is an integer in the range [-4, 4],} \\ 0, & \text{otherwise} \end{cases}$$

Discrete Uniform Random Variable

$$\mathbf{E}[X] = \sum_x x p_X(x) = \frac{1}{9} \sum_{x=-4}^{4} x = 0$$

$$\text{var}(X) = \mathbf{E}\left[(x - \mathbf{E}[x])^2\right] = \sum_x (x - \mathbf{E}[x])^2 p_X(x) = \frac{1}{9} \sum_{x=-4}^{4} x^2 = \frac{60}{9}$$

$$\text{Or, let } Z = (X - \mathbf{E}[X])^2 = X^2$$

$$\Rightarrow p_Z(\mathrm{z}) = \begin{cases} 2/9, & \text{if } z = 1,4,9,16 \\ 1/9, & \text{if } z = 0 \\ 0, & \text{otherwise} \end{cases}$$

$$\text{var}(X) = \mathbf{E}[Z] = \sum_z z p_Z(\mathrm{z}) = \frac{60}{9}$$

Properties of Mean and Variance (1/2)

- Let X be a random variable and let

$$Y = aX + b$$

a linear function of X

where a and b are given scalars

Then,

$$\mathbf{E}[Y] = a\mathbf{E}[X] + b$$

$$\text{var}(Y) = a^2 \text{ var}(X)$$

- If $g(X)$ is a linear function of X, then

$$\mathbf{E}[g(X)] = g(\mathbf{E}[X])$$

How to verify it ?

Properties of Mean and Variance (2/2)

$$\mathbf{E}[Y] = \sum_x (ax+b)p_X(x) = \left[a\sum_x x p_X(x)\right] + \left[b\underbrace{\sum_x p_X(x)}_{1}\right] = a\mathbf{E}[X] + b$$

$$\begin{aligned}
\mathrm{var}(Y) &= \sum_x (ax+b-\mathbf{E}[aX+b])^2 p_X(x) \\
&= \sum_x (ax+b-a\mathbf{E}[X]-b)^2 p_X(x) \\
&= a^2 \underbrace{\sum_x (x-\mathbf{E}[X])^2 p_X(x)}_{\mathrm{var}(X)} \\
&= a^2\,\mathrm{var}(X)
\end{aligned}$$

Variance in Terms of Moments Expression

- We can also express variance of a random variable X as

$$\operatorname{var}(X) = \mathbf{E}\left[X^2\right] - \left(\mathbf{E}[X]\right)^2$$

$$\begin{aligned}
\operatorname{var}(X) &= \sum_x \left(x - \mathbf{E}[X]\right)^2 p_X(x) \\
&= \sum_x \left(x^2 - 2x\mathbf{E}[X] + \left(\mathbf{E}[X]\right)^2\right) p_X(x) \\
&= \left[\sum_x x^2 p_X(x)\right] + 2\mathbf{E}[X]\left[\underbrace{\sum_x x p_X(x)}_{1}\right] + \left(\mathbf{E}[X]\right)^2 \left[\underbrace{\sum_x x p_X(x)}_{1}\right] \\
&= \mathbf{E}\left[X^2\right] - 2\left(\mathbf{E}[X]\right)^2 + \left(\mathbf{E}[X]\right)^2 \\
&= \mathbf{E}\left[X^2\right] - \left(\mathbf{E}[X]\right)^2
\end{aligned}$$

Mean and Variance of the **Bernoulli**

- **Example 2.5.** Consider the experiment of tossing a biased coin, which comes up a head with probability p and a tail with probability $1-p$, and the **Bernoulli** random variable with PMF

$$p_X(x)=\begin{cases} p, & \text{if } x=1 \\ 1-p, & \text{if } x=0 \end{cases}$$

$$\mathbf{E}[X]=\sum_x x p_X(x)=1\cdot p+0\cdot(1-p)=p$$

$$\mathbf{E}\left[X^2\right]=\sum_x x^2 p_X(x)=1^2\cdot p+0^2\cdot(1-p)=p$$

$$\operatorname{var}(X)=\mathbf{E}\left[X^2\right]-(\mathbf{E}[X])^2=p-p^2=p(1-p)$$

Mean and Variance of the **Discrete Uniform**

- Consider a **discrete uniform** random variable with a nonzero PMF in the range [a, b]

$$p_X(x) = \begin{cases} \dfrac{1}{b-a+1}, & \text{if } x = a, a+1, \ldots, b \\ 0, & \text{otherwise} \end{cases}$$

$$\mathbf{E}[X] = \sum_x x p_X(x) = \frac{1}{b-a+1} \sum_a^b x = \boxed{\frac{a+b}{2}}$$

$$\mathbf{E}\left[X^2\right] = \frac{1}{b-a+1} \sum_a^b x^2 = \frac{1}{b-a+1} \cdot \left(\frac{b(b+1)(2b+1)}{6} - \frac{(a-1)(a)(2a-1)}{6} \right)$$

$$\operatorname{var}(X) = \mathbf{E}\left[X^2\right] - (\mathbf{E}[X])^2 = \frac{1}{b-a+1} \cdot \left(\frac{b(b+1)(2b+1)}{6} - \frac{(a-1)(a)(2a-1)}{6} \right) - \left(\frac{a+b}{2} \right)^2$$

$$= \frac{1}{b-a+1} \cdot \frac{(b-a)(b-a+1)(b-a+2)}{12} = \boxed{\frac{(b-a)(b-a+2)}{12}}$$

Mean and Variance of the **Poisson**

- Consider a **Poisson** random variable with a PMF

$$p_X(x) = e^{-\lambda} \frac{\lambda^x}{x!}, \quad x = 0,1,2,\ldots,$$

$$\mathbf{E}[X] = \sum_x x p_X(x) = \sum_{x=0}^{\infty} x e^{-\lambda} \frac{\lambda^x}{x!} = \lambda \sum_{x=1}^{\infty} e^{-\lambda} \frac{\lambda^{x-1}}{(x-1)!} = \lambda \underbrace{\sum_{x'=0}^{\infty} e^{-\lambda} \frac{\lambda^{x'}}{x'!}}_{1} = \lambda$$

$$\mathbf{E}[X^2] = \sum_x x^2 p_X(x) = \sum_{x=0}^{\infty} x^2 e^{-\lambda} \frac{\lambda^x}{x!} = \lambda \sum_{x=1}^{\infty} x \frac{\lambda^{x-1}}{(x-1)!}$$

$$= \lambda \sum_{x'=0}^{\infty} (x'+1) e^{-\lambda} \frac{\lambda^{x'}}{x'!} = \lambda \left[\left(\sum_{x'=0}^{\infty} x' e^{-\lambda} \frac{\lambda^{x'}}{x'!} \right) + \left(\underbrace{\sum_{x'=0}^{\infty} e^{-\lambda} \frac{\lambda^{x'}}{x'!}}_{1} \right) \right] = \lambda(\mathbf{E}[X] + 1) = \lambda^2 + \lambda$$

$$\mathrm{var}(X) = \mathbf{E}[X^2] - (\mathbf{E}[X])^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$$

Mean and Variance of the **Binomial**

- Consider a **binomial** random variable with a PMF

$$p_X(x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0,1,\ldots,n$$

$$\mathbf{E}[X] = \sum_x x p_X(x) = \sum_{x=0}^{n} x \binom{n}{x} p^x (1-p)^{n-x} = \sum_{x=1}^{n} x \binom{n}{x} p^x (1-p)^{n-x} = \sum_{x=1}^{\infty} x \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

$$= np \sum_{x=1}^{n} \frac{(n-1)!}{(x-1)!(n-x)!} p^{x-1} (1-p)^{n-x} = np \underbrace{\sum_{x'=0}^{n-1} \frac{(n-1)!}{x'!(n-x')!} p^{x'} (1-p)^{n-x'}}_{1} = \boxed{np}$$

$$\mathbf{E}[X^2] = \mathbf{E}[X^2 - X] + \mathbf{E}[X] \text{ (to be verified later on!)}$$

$$\mathbf{E}[X^2 - X] = \mathbf{E}[X(X-1)] = \sum_{x=0}^{n} x(x-1) \binom{n}{x} p^x (1-p)^{n-x} = \sum_{x=2}^{n} x(x-1) \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

$$= n(n-1)p^2 \underbrace{\sum_{x=2}^{n} \frac{(n-2)!}{(x-2)!(n-x)!} p^{x-2} (1-p)^{n-x}}_{1} = n(n-1)p^2$$

$$\operatorname{var}(X) = \mathbf{E}[X^2] - (\mathbf{E}[X])^2 = \mathbf{E}[X^2 - X] + \mathbf{E}[X] - (\mathbf{E}[X])^2$$

$$= n(n-1)p^2 + np - n^2p^2 = \boxed{np(1-p)}$$

Mean and Variance of the **Geometric**

- Consider a **geometric** random variable with a PMF

$$p_X(x) = (1-p)^{x-1} p, \quad x = 1,2,\ldots,$$

$$\mathbf{E}[X] = \sum_x x p_X(x) = \sum_{x=0}^{\infty} x(1-p)^{x-1} p = p \sum_{x=1}^{\infty} x q^{x-1} \quad (\text{let } q = 1\text{-}p < 1)$$

$$= p \frac{d\left(\sum_{x=1}^{\infty} q^x\right)}{dq} = p \frac{d\left(\frac{1}{1-q}\right)}{dq} = p \frac{1}{(1-q)^2} = \boxed{\frac{1}{p}}$$

$$\mathbf{E}[X^2] = \mathbf{E}[X^2 - X] + \mathbf{E}[X] \ (\text{to be verified later on!})$$

$$\mathbf{E}[X^2 - X] = \mathbf{E}[X(X-1)] = \sum_{x=0}^{\infty} x(x-1)(1-p)^{x-1} p = pq \sum_{x=2}^{\infty} x(x-1) q^{x-2} p \quad (\text{let } q = 1\text{-}p < 1)$$

$$= pq \sum_{x=2}^{\infty} x(x-1) q^{x-2} p = pq \frac{d^2\left(\frac{1}{1-q}\right)}{d^2 q} = pq \frac{2}{(1-q)^3} = \frac{2(1\text{-}p)}{p^2}$$

$$\operatorname{var}(X) = \mathbf{E}[X^2] - (\mathbf{E}[X])^2 = \mathbf{E}[X^2 - X] + \mathbf{E}[X] - (\mathbf{E}[X])^2$$

$$= \frac{2(1\text{-}p)}{p^2} + \frac{1}{p} - \frac{1}{p^2} = \boxed{\frac{(1-p)}{p^2}}$$

Recitation

- SECTION 2.4 Expectation, Mean, Variance
  - Problems 18, 19, 21, 24