# Multivariate Methods

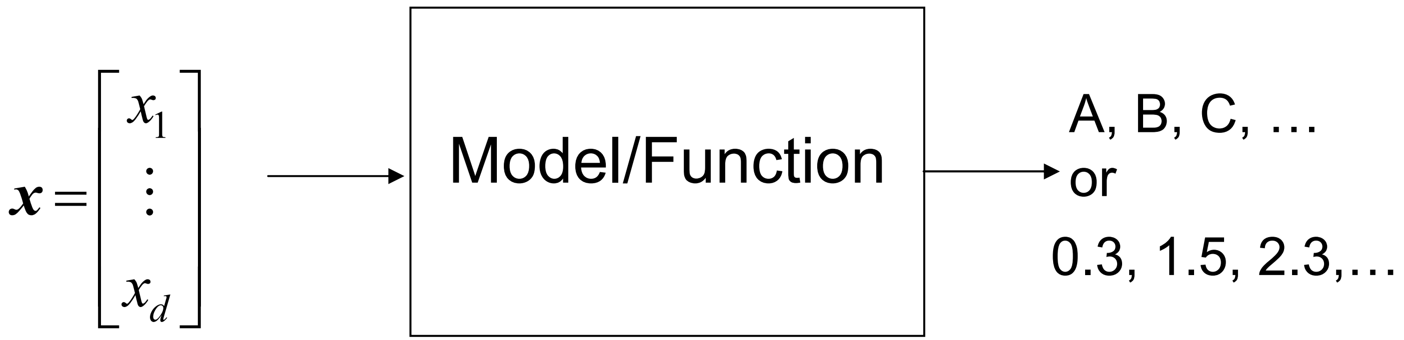
Berlin Chen, 2005

References:

1. ***Introduction to Machine Learning*** **, Chapter 5**

# Multivariate Methods

- Input: a data sample with multiple features (variables/ inputs)
- Output (of Prediction)
  - Classification: class code (discrete variable)
  - Regression: real number (continuous variable)




- Supervised learning
  - Model/function to be trained with labeled training samples

# Multivariate Data

- Each data sample is represented by an observation vector with $d$ dimensions
  - Each dimension of the vector is termed input/feature/attribute
  - Dimensions with different types and value domains
- The whole data set of size $N$ can be viewed as a data matrix

$$\boldsymbol{X} = \begin{bmatrix} X_1^1 & X_2^1 & \cdots & X_d^1 \\ X_1^2 & X_2^2 & \cdots & X_d^2 \\ \vdots & \vdots & \vdots & \vdots \\ X_1^N & X_2^N & \cdots & X_d^N \end{bmatrix}$$

samples/instances/items

Inputs/features/attributes

Reduced to a naïve Bayes' classifier

  - Features are usually assumed correlated !

$$P(\boldsymbol{x}|C_i) = P(x_1, x_2, \cdots, x_d | C_i) \overset{?}{=} \prod_{j=1}^{d} P(x_j | C_i)$$

    - Otherwise, there is no need for a multivariate analysis

# Multivariate Data (cont.)

- Motivations for multivariate data analysis
  - Simplification
    - Assume that the large body of data can be well summarized by means of relatively few parameters
  - Exploration
    - Predict the value of one variable form the values of other variables
      - Multivariate classification (Discrete)
      - Multivariate regression (Numeric)

# Parameter Estimation

- Mean of data samples

$$E[\boldsymbol{X}] = \int \boldsymbol{x} \cdot P(\boldsymbol{x}) dx = \boldsymbol{\mu} = \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_d \end{bmatrix}$$

$\mu_1$ ← Mean of Column 1 of matrix ***X***

$\mu_d$ ← Mean of Column *d* of matrix ***X***

- Covariance matrix of data samples

$$\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1d} \\ \sigma_{21} & \sigma_2^2 & \cdots & \sigma_{2d} \\ \vdots & \vdots & \vdots & \vdots \\ \sigma_{d1} & \cdots & \cdots & \sigma_d^2 \end{bmatrix}$$

$\sigma_i^2$ : variance of a variable $X_i$

$\sigma_{ij}$ : covariace of two variables $X_i$ and $X_j$

$$E[X_i X_j] = \int_{x_i} \int_{x_j} x_i x_j P(x_i x_j) dx_i dx_j$$

$$\sigma_{ij} \equiv \mathrm{Cov}(X_i, X_j) = E[(X_i - \mu_i)(X_j - \mu_j)] = E[X_i X_j] - \mu_i \mu_j$$

if $X_i$ and $X_j$ are independent

$$E[X_i X_j] = E[X_i] E[X_j] = \mu_i \mu_j$$

$$\sigma_{ii} \equiv \mathrm{Cov}(X_i, X_i) = E[(X_i - \mu_i)(X_i - \mu_i)] = E[(X_i)^2] - (\mu_i)^2 = \sigma_i^2$$

# Parameter Estimation (cont.)

- Covariance matrix is symmetric
  - Diagonal terms: variances
  - Off-diagonal terms: covariances

$$\Sigma = \mathrm{Cov}(\boldsymbol{X}) = E\left[(\boldsymbol{X}-\boldsymbol{\mu})(\boldsymbol{X}-\boldsymbol{\mu})^T\right] = E\left[\boldsymbol{X}\boldsymbol{X}^T\right] - \boldsymbol{\mu}\boldsymbol{\mu}^T$$

- Correlation between two variables $X_i$ and $X_j$

$$\mathrm{Corr}(X_i, X_j) = \frac{\sigma_{ij}}{\sigma_i \sigma_j}, \qquad -1 \le \mathrm{Corr}(X_i, X_j) \le 1$$

independent $\rightarrow$ uncorrelated

  - Two variables independent $\rightarrow$ covariance =correlation=0
  - But covariance =correlation=0 does not imply two variables are independent (nonlinear dependence)

# Parameter Estimation (cont.)

- If $X_i$ and $X_j$ are linear dependent

$$X_j = a \cdot X_i + b \quad (a > 0)$$
$$\Rightarrow E[X_j] = \mu_j = a \cdot [X_i] + b = a \cdot \mu_i + b$$
$$\sigma_j^2 = \text{var}(X_j) = E\left[(X_j - \mu_j)^2\right] = a^2 \cdot \text{var}(X_i) = a^2 \cdot \sigma_i^2$$

$$\sigma_{ij} = E\left[(X_i - \mu_i)(X_j - \mu_j)\right] = E\left[(X_i - \mu_i)((a \cdot X_i + b) - (a \cdot \mu_i + b))\right] = a \cdot E\left[(X_j - \mu_j)^2\right] = a \cdot \sigma_i^2$$

$$\Rightarrow \rho_{ij} = \frac{\sigma_{ij}}{\sigma_i \sigma_j} = \frac{a \cdot \sigma_i^2}{\sigma_i \cdot (a \cdot \sigma_i)} = 1$$

Similarly, for

$$X_j = -a \cdot X_i + b \quad (a > 0)$$
$$\rho_{ij} = -1$$

# Parameter Estimation (cont.)

- Maximum Likelihood Estimators
  - Sample mean as an estimator for mean

$$\bar{\boldsymbol{\mu}} = \boldsymbol{m} = \frac{\sum_{t=1}^{N} \boldsymbol{x}^t}{N} \text{ with } m_i = \frac{\sum_{t=1}^{N} x_i^t}{N}, \text{ i} = 1, \ldots, d$$

  - Sample covariance matrix as an estimator for covariance matrix

$$s_i^2 = \frac{\sum_{t=1}^{N} \left(x_i^t - m_i\right)^2}{N}$$

$$s_{ij} = \frac{\sum_{t=1}^{N} \left(x_i^t - m_i\right)\left(x_j^t - m_j\right)}{N}$$

$$\bar{\boldsymbol{\Sigma}} = \boldsymbol{C} = \begin{bmatrix} s_1^2 & s_{12} & \cdots & s_{1d} \\ s_{21} & s_2^2 & \cdots & s_{2d} \\ \vdots & \vdots & \vdots & \vdots \\ s_{d1} & \cdots & \cdots & s_d^2 \end{bmatrix}$$

  - Sample correlation coefficients

$$r_{ij} = \frac{s_{ij}}{s_i s_j}$$

# Multivariate Normal Distribution

- A random variable $\boldsymbol{x}$ that is $d$-dimensional and normal distributed, is denoted as $\boldsymbol{x} \sim N_d(\boldsymbol{\mu}, \boldsymbol{\Sigma})$

$$P(\boldsymbol{x}) = \frac{1}{(2\pi)^{d/2} |\boldsymbol{\Sigma}|^{1/2}} \exp\left[ -\frac{1}{2} (\boldsymbol{x} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1} (\boldsymbol{x} - \boldsymbol{\mu}) \right]$$

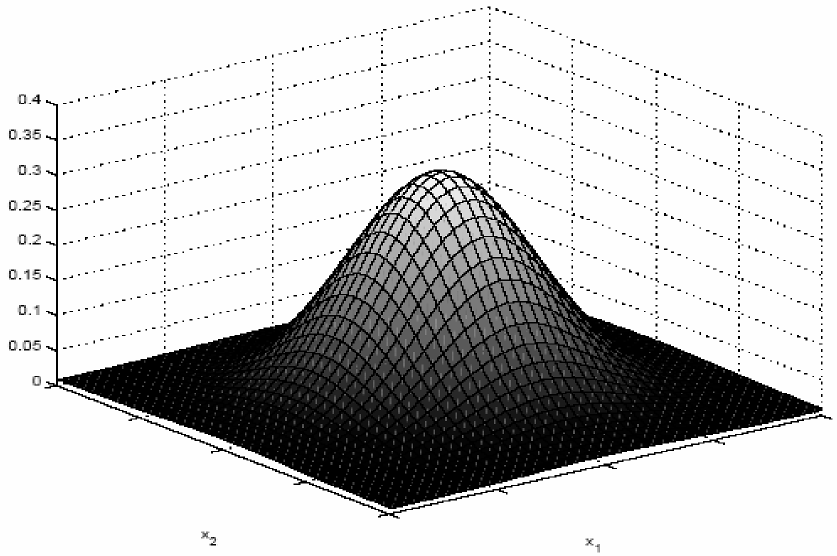
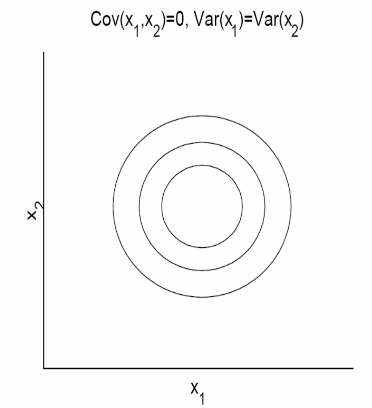


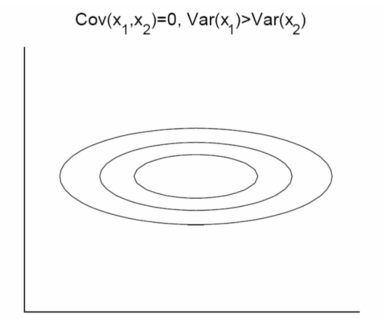

Figure 5.1: Bivariate normal distribution.

# Multivariate Normal Distribution (cont.)

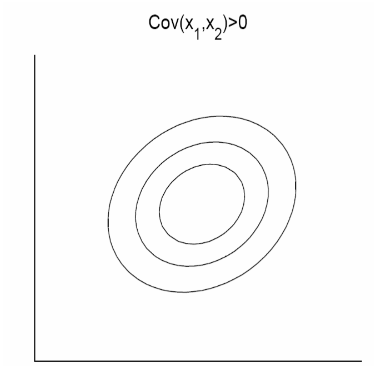
- If each dimension is independent of one another and with same variance value



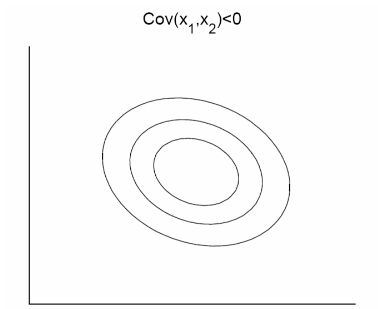

- If each dimension is independent of one another and with different variance value




- If each dimension is dependent of one another with $\text{Corr}(X_1, X_2) > 0$




- If each dimension is dependent of one another with $\text{Corr}(X_1, X_2) < 0$



# Multivariate Normal Distribution (cont.)

- If the components of random variable $\boldsymbol{x}$ are independent

$$P(\boldsymbol{x}) = P(x_1, \cdots, x_j, \cdots, x_d) = \prod_{j=1}^{d} P(x_j)$$

$$= \frac{1}{(2\pi)^{d/2} \prod_{j=1}^{d} \sigma_j} \exp\left[ -\frac{1}{2} \sum_{j=1}^{d} \left( \frac{x_j - \mu_j}{\sigma_j} \right)^2 \right]$$

Also note that $P(x_j) = \dfrac{1}{(2\pi)^{1/2} \sigma_j} \exp\left[ -\dfrac{1}{2} \left( \dfrac{x_j - \mu_j}{\sigma_j} \right)^2 \right]$

# Multivariate Normal Distribution (cont.)

- Recall: The project of a *d*-dimensional normal distribution on a vector $\boldsymbol{w}$ is univariate normal (suppose that $\|\boldsymbol{w}\|=1$ )

$$\boldsymbol{w}^T\boldsymbol{x} = w_1x_1 + w_2x_2 + \cdots + w_dx_d \sim N\left(\boldsymbol{w}^T\boldsymbol{\mu}, \boldsymbol{w}^T\boldsymbol{\Sigma}\boldsymbol{w}\right)$$

$$E\left[\boldsymbol{w}^T\boldsymbol{x}\right] = \boldsymbol{w}^T E[\boldsymbol{x}] = \underbrace{\boldsymbol{w}^T\boldsymbol{\mu}}_{\text{scalar}}$$

$$\mathrm{Var}\left(\boldsymbol{w}^T\boldsymbol{x}\right) = E\left[\left(\boldsymbol{w}^T\boldsymbol{x} - \boldsymbol{w}^T\boldsymbol{\mu}\right)\left(\boldsymbol{w}^T\boldsymbol{x} - \boldsymbol{w}^T\boldsymbol{\mu}\right)^T\right]$$

$$= E\left[\boldsymbol{w}^T(\boldsymbol{x}-\boldsymbol{\mu})(\boldsymbol{x}-\boldsymbol{\mu})^T\boldsymbol{w}\right] = \boldsymbol{w}^T E\left[(\boldsymbol{x}-\boldsymbol{\mu})(\boldsymbol{x}-\boldsymbol{\mu})^T\right]\boldsymbol{w} = \underbrace{\boldsymbol{w}^T\boldsymbol{\Sigma}\boldsymbol{w}}_{\text{scalar}}$$

- The project of a *d*-dimensional normal distribution to a *k*-dimensional space is *k*-variate normal

$$\boldsymbol{W}^T\boldsymbol{x} \sim N_k\left(\underbrace{\boldsymbol{W}^T\boldsymbol{\mu}}_{k\text{-dim vector}}, \underbrace{\boldsymbol{W}^T\boldsymbol{\Sigma}\boldsymbol{W}}_{k\text{x}k\text{ matrix}}\right) \qquad \boldsymbol{W} = \begin{bmatrix}\boldsymbol{w}_1 & \boldsymbol{w}_2 & \cdots & \boldsymbol{w}_k\end{bmatrix}$$

# Multivariate Classification

- Normal density $N_d(\boldsymbol{\mu}_i, \boldsymbol{\Sigma}_i)$ as the class-conditional probability $P(\boldsymbol{x}|C_i)$ of random variable $\boldsymbol{x} \in \Re^d$

$$P(\boldsymbol{x}|C_i) = \frac{1}{(2\pi)^{d/2}|\boldsymbol{\Sigma}_i|^{1/2}} \exp\left[-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu}_i)^T \boldsymbol{\Sigma}_i^{-1}(\boldsymbol{x}-\boldsymbol{\mu}_i)\right]$$

- Define the discriminant function as

$$g_i(\boldsymbol{x}) = \log P(\boldsymbol{x}|C_i) + \log P(C_i)$$
$$= -\frac{d}{2}\log 2\pi - \frac{1}{2}\log|\boldsymbol{\Sigma}_i| - \frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu}_i)^T \boldsymbol{\Sigma}_i^{-1}(\boldsymbol{x}-\boldsymbol{\mu}_i) + \log P(C_i)$$

# Multivariate Classification (cont.)

- Maximum likelihood (ML) training of classifiers
  - Given a set of labeled samples

$$\boldsymbol{X} = \{\boldsymbol{x}^t, \boldsymbol{r}^t\}, \; r_i^t = 1 \text{ if } \boldsymbol{x}^t \in C_i \text{ and } 0 \text{ otherwise}$$

$$\hat{P}(C_i) = \frac{\sum_{t=1}^{N} r_i^t}{N}$$

$$\hat{\boldsymbol{\mu}}_i = \boldsymbol{m}_i \text{ (sample mean)} = \frac{\sum_{t=1}^{N} r_i^t \boldsymbol{x}^t}{\sum_{t=1}^{N} r_i^t}$$

$$\hat{\boldsymbol{\Sigma}}_i = \boldsymbol{S}_i \text{ (sample covariance matrix)} = \frac{\sum_{t=1}^{N} r_i^t (\boldsymbol{x}^t - \boldsymbol{m}_i)(\boldsymbol{x}^t - \boldsymbol{m}_i)^T}{\sum_{t=1}^{N} r_i^t}$$

# Quadratic Discriminant

- The discriminant function with normal class-conditional density can be expressed as a quadratic discriminant

$$
\begin{aligned}
g_i(\boldsymbol{x}) &= \log P(\boldsymbol{x}|C_i) + \log P(C_i) \\
&= -\frac{d}{2}\log 2\pi - \frac{1}{2}\log|\boldsymbol{S}_i| - \frac{1}{2}(\boldsymbol{x}-\boldsymbol{m}_i)^T \boldsymbol{S}_i^{-1}(\boldsymbol{x}-\boldsymbol{m}_i) + \log P(C_i) \\
&= -\frac{1}{2}\left(\boldsymbol{x}^T \boldsymbol{S}_i^{-1}\boldsymbol{x} - 2\boldsymbol{x}^T \boldsymbol{S}_i^{-1}\boldsymbol{m}_i + \boldsymbol{m}_i^T \boldsymbol{S}_i^{-1}\boldsymbol{m}_i\right) - \frac{d}{2}\log 2\pi - \frac{1}{2}\log|\boldsymbol{S}_i| + \log P(C_i) \\
&= \boldsymbol{x}^T \boldsymbol{W}_i \boldsymbol{x} + \boldsymbol{w}_i^T \boldsymbol{x} + w_{i0}
\end{aligned}
$$

quadratic discriminant

Recall : second - order polynomial for scalar variable

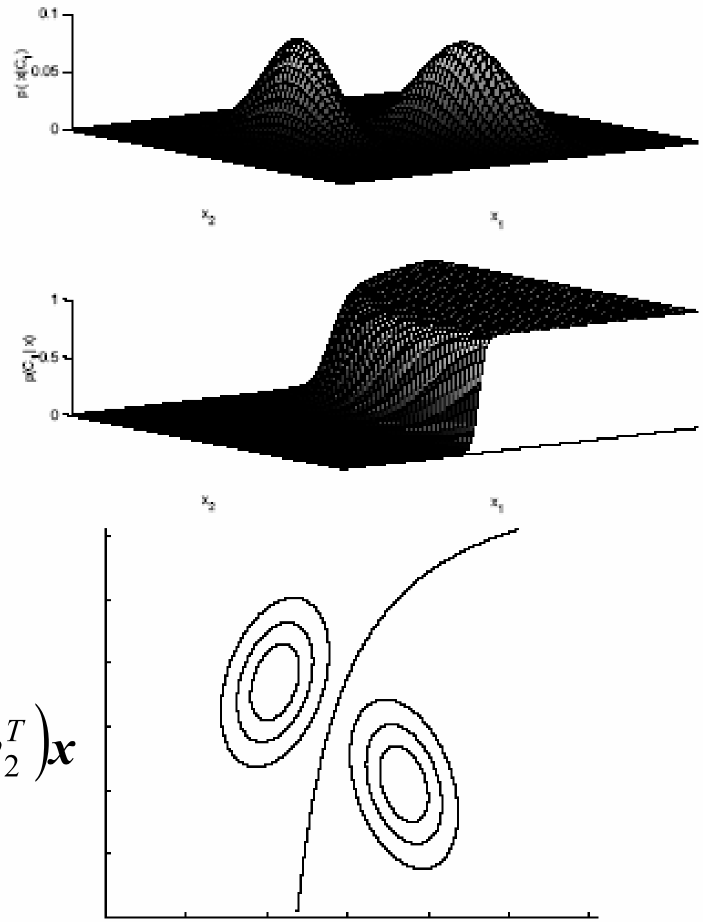
$g_i(x) = w_{i2}x^2 + w_{i1}x + w_{i0}$

$$\boldsymbol{W}_i = -\frac{1}{2}\boldsymbol{S}_i^{-1}$$

$$w_i = \boldsymbol{S}_i^{-1}\boldsymbol{m}_i$$

$$w_{i0} = -\frac{d}{2}\log 2\pi - \frac{1}{2}\boldsymbol{m}_i^T \boldsymbol{S}_i^{-1}\boldsymbol{m}_i - \frac{1}{2}\log|\boldsymbol{S}_i| + \log P(C_i)$$

# Quadratic Discriminant (cont.)



likelihood

posterior

likelihood

$$g_1(\boldsymbol{x}) = \boldsymbol{x}^T \boldsymbol{W}_1 \boldsymbol{x} + \boldsymbol{w}_1^T \boldsymbol{x} + w_{10}$$

$$g_2(\boldsymbol{x}) = \boldsymbol{x}^T \boldsymbol{W}_2 \boldsymbol{x} + \boldsymbol{w}_2^T \boldsymbol{x} + w_{20}$$

$$\begin{aligned} \tilde{g}(\boldsymbol{x}) &= g_1(\boldsymbol{x}) - g_2(\boldsymbol{x}) \\ &= \boldsymbol{x}^T (\boldsymbol{W}_1 - \boldsymbol{W}_2)\boldsymbol{x} + \left(\boldsymbol{w}_1^T - \boldsymbol{w}_2^T\right)\boldsymbol{x} \\ &\quad + (w_{10} - w_{20}) \\ &= \boldsymbol{x}^T W' \boldsymbol{x} + \boldsymbol{w}' \boldsymbol{x} + w_0' \end{aligned}$$

Figure 5.3: Classes have different covariance matrices.

# Linear Discriminant

- The discriminant function with normal class-conditional density sharing the same covariance matrix can be expressed as a linear discriminant

$$\boldsymbol{S} = \sum_{i=1}^{K} \hat{P}(C_i) \cdot \boldsymbol{S}_i$$

All classes share the same covariance matrix

$$g_i(\boldsymbol{x}) = \log P(\boldsymbol{x}|C_i) + \log P(C_i)$$

$$= -\frac{1}{2}\left(\boldsymbol{x}^T \boldsymbol{S}^{-1}\boldsymbol{x} - 2\boldsymbol{x}^T \boldsymbol{S}^{-1}\boldsymbol{m}_i + \boldsymbol{m}_i^T \boldsymbol{S}^{-1}\boldsymbol{m}_i\right) - \frac{d}{2}\log 2\pi - \frac{1}{2}\log|\boldsymbol{S}| + \log P(C_i)$$

$$\Rightarrow \tilde{g}_i(\boldsymbol{x}) = \boldsymbol{w}_i^T \boldsymbol{x} + w_{i0}$$

$$w_i = \boldsymbol{S}^{-1}\boldsymbol{m}_i$$

$$w_{i0} = -\frac{d}{2}\log 2\pi - \frac{1}{2}\boldsymbol{m}_i^T \boldsymbol{S}^{-1}\boldsymbol{m}_i - \frac{1}{2}\log|\boldsymbol{S}| + \log P(C_i)$$

# Linear Discriminant (cont.)

$$g_1(\boldsymbol{x}) = \boldsymbol{w}_1^T \boldsymbol{x} + w_{10}$$

$$g_2(\boldsymbol{x}) = \boldsymbol{w}_2^T \boldsymbol{x} + w_{20}$$

$$\tilde{g}(\boldsymbol{x}) = g_1(\boldsymbol{x}) - g_2(\boldsymbol{x})$$
$$= \left(\boldsymbol{w}_1^T - \boldsymbol{w}_2^T\right)\boldsymbol{x} + \left(w_{10} - w_{20}\right)$$
$$= \boldsymbol{w}'^T \boldsymbol{x} + w_0'$$

covariances (off-diagonal terms of covariance matrix) are not equal to zero

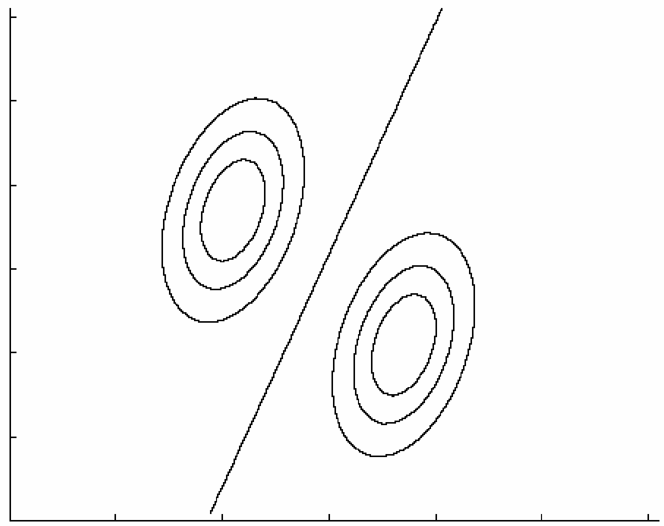


Figure 5.4: Covariances may be arbitary but shared by both classes.

**kelihood**

# Naïve Bayes' Classifier

- The discriminant function with normal class-conditional density sharing the same diagonal covariance matrix can be expressed as a naïve Bayes' classifier

$$g_i(\boldsymbol{x}) = -\frac{1}{2}\sum_{j=1}^{d}\left(\frac{x_j - m_{ij}}{s_j}\right)^2 + \log P(C_i)$$

variances (diagonal terms of covariance matrix) are not equal

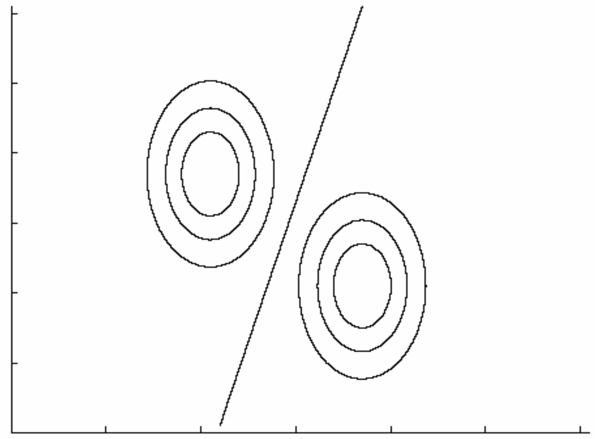


Figure 5.5: All classes have equal, diagonal covariance matrices but variances are not equal.

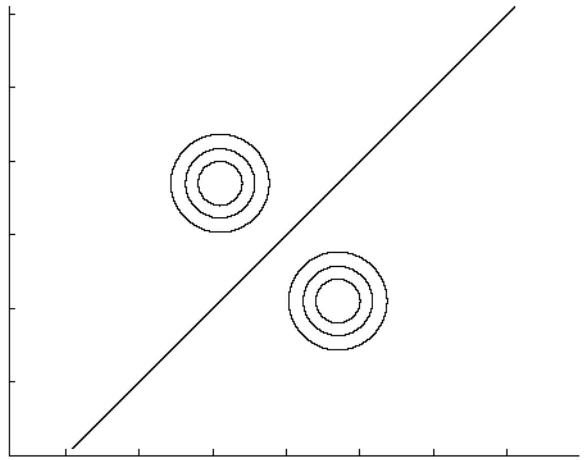
**likelihood**

# Naïve Bayes' Classifier (cont.)

- If the variances (diagonal terms) of the naïve Bayes' classifier are further set to equal for all dimensions

$$g_i(\boldsymbol{x}) = -\frac{1}{2s^2}\sum_{j=1}^{d}\left(x_j - m_{ij}\right)^2 + \log P(C_i)$$

The Mahalanobis distance is reduced to Euclidean distance.
(all variables/features have the same variance and are independent)



**likelihood**

Figure 5.6: All classes have equal, diagonal covariance matrices of equal variances on both dimensions.

# Nearest Mean Classifier

- Assign the data sample to the class of the nearest mean
  - If the priors $P(C_i)$ are further set to equal

$$g_i(\boldsymbol{x}) = -\sum_{j=1}^{d}(x_j - m_{ij})^2 = -\|\boldsymbol{x} - \boldsymbol{m}_i\|^2$$

$$= -(\boldsymbol{x} - \boldsymbol{m}_i)^T(\boldsymbol{x} - \boldsymbol{m}_i)$$

$$= -(\boldsymbol{x}^T\boldsymbol{x} - 2\boldsymbol{m}_i^T\boldsymbol{x} + \boldsymbol{m}_i^T\boldsymbol{m}_i)$$

$$\Rightarrow \bar{g}_i(\boldsymbol{x}) = \boldsymbol{m}_i^T\boldsymbol{x} - \frac{1}{2}\boldsymbol{m}_i^T\boldsymbol{m}_i$$

$$= \boldsymbol{w}_i^T\boldsymbol{x} + w_{i0}$$

where $\boldsymbol{w}_i = \boldsymbol{m}_i$

$$w_{i0} = -\frac{1}{2}\boldsymbol{m}_i^T\boldsymbol{m}_i = -\frac{1}{2}\|\boldsymbol{m}_i\|^2$$

# Tuning Complexity

- Tradeoff between the bias and variance of an estimator

**Table 5.1** Reducing variance through simplifying assumptions.

| Assumption | Covariance matrix | No. of parameters |
|---|---|---|
| Shared, Hyperspheric | $\mathbf{S}_i = \mathbf{S} = s^2\mathbf{I}$ | 1 |
| Shared, Axis-aligned | $\mathbf{S}_i = \mathbf{S}$, with $s_{ij} = 0$ | $d$ |
| Shared, Hyperellipsoidal | $\mathbf{S}_i = \mathbf{S}$ | $d(d+1)/2$ |
| Different, Hyperellipsoidal | $\mathbf{S}_i$ | $K \cdot (d(d+1)/2)$ |

  – Simplifying covariance matrix → decreasing number of parameters → introducing estimation bias
  – Arbitrary covariance matrix → much more data is needed → introducing estimation variance

- Regularized Discriminant Analysis (RDA, 1989)
  – Use a weighted average of three special cases of covariance matrix

$$\boldsymbol{S}' = \alpha \cdot \sigma^2 \boldsymbol{I} + \beta \cdot \boldsymbol{S} + (1 - \alpha - \beta) \cdot \boldsymbol{S}_i$$

a shared diagonal covariance matrix (with equal variance) for all classes

a shared covariance matrix for all classes

a specific covariance matrix for each class

# Discrete Features

- Features that take one of *n* different values
  - E.g. $\text{pixel} \in \{\text{on}, \text{off}\}; \text{color} \in \{\text{red}, \text{blue}, \text{green}, \text{black}\}$
- If each feature is a binary random variable $x_j^t$ $\left(x_j^t = 1 \text{ or } x_j^t = 0\right)$
  - Bernoulli distribution

$$P_{ij} = P\left(x_j^t = 1 \middle| C_i\right) \quad 1 - P_{ij} = P\left(x_j^t = 0 \middle| C_i\right) \quad \Rightarrow P\left(x_j^t \middle| C_i\right) = \left(P_{ij}\right)^{x_j^t}\left(1 - P_{ij}\right)^{\left(1 - x_j^t\right)}$$

  - If features are further assumed to be independent
    - Assumption for Naïve Bayes' classifier $\boldsymbol{x}^t = \left\{x_1^t, \cdots, x_j^t, \cdots, x_d^t\right\}$

$$P\left(\boldsymbol{x}^t \middle| C_i\right) = P\left(x_1^t, \cdots, x_d^t \middle| C_i\right) = \prod_{j=1}^{d} P\left(x_j^t \middle| C_i\right) = \prod_{j=1}^{d} \left(P_{ij}\right)^{x_j^t}\left(1 - P_{ij}\right)^{\left(1 - x_j^t\right)}$$

    - Discriimant function

$$g_i\left(\boldsymbol{x}^t\right) = \log P\left(\boldsymbol{x}^t \middle| C_i\right) + \log P\left(C_i\right) = \sum_{j=1}^{d}\left[x_j^t \log P_{ij} + \left(1 - x_j^t\right)\log\left(1 - P_{ij}\right)\right] + \log P\left(C_i\right)$$

– Linear ?

# Discrete Features (cont.)

- Appendix A

$$g_i(\boldsymbol{x}^t) = \sum_{j=1}^{d}\left[x_j^t \log P_{ij} + (1-x_j^t)\log(1-P_{ij})\right] + \log P(C_i)$$

$$= \left[\sum_{j=1}^{d} x_j^t \log P_{ij}\right] - \left[\sum_{j=1}^{d} x_j^t (1-\log P_{ij})\right] + \left[\sum_{j=1}^{d} (1-\log P_{ij})\right] + \log P(C_i)$$

$$= \left[\sum_{j=1}^{d} x_j^t \left(2\cdot(\log P_{ij})-1\right)\right] + \left[\sum_{j=1}^{d} (1-\log P_{ij})\right] + \log P(C_i)$$

$$= \boldsymbol{w}_i^T \boldsymbol{x}^t + D_i$$ **Linear**

where

$$\boldsymbol{w}_i = \begin{bmatrix} 2\cdot(\log P_{i1})-1 \\ 2\cdot(\log P_{i2})-1 \\ \vdots \\ 2\cdot(\log P_{id})-1 \end{bmatrix} \qquad D_i = \left[\sum_{j=1}^{d} (1-\log P_{ij})\right] + \log P(C_i)$$

# Discrete Features (cont.)

- Maximum likelihood estimation for $P_{ij}$ (for binary variable)

$$\hat{P}_{ij} = \frac{\sum_{t=1}^{N} x_j^t \cdot r_i^t}{\sum_{t=1}^{N} r_i^t}$$

- Extension: features are independent multinomial random variables

$$x_j^t \in \{v_{j1}, \cdots, v_{jk}, \cdots, v_{jn_j}\} \qquad z_{jk}^t = \begin{cases} 1 & \text{if } x_j^t = v_{jk} \\ 0 & \text{otherwise} \end{cases}$$

  - Define the probability that $x_j^t$ belongs to $C_i$ and take $v_{jk}$ :

$$P_{ijk} = P\left(z_{jk}^t = 1 \middle| C_i\right) = P\left(x_j^t = v_{jk} \middle| C_i\right) \Rightarrow \prod_{j=1}^{d} P\left(x_j^t \middle| C_i\right) = \prod_{k=1}^{n_j} P_{ijk}^{z_{jk}^t}$$

# Discrete Features (cont.)

- Assumption for Naïve Bayes' classifier

$$P\left(\boldsymbol{x}^t \middle| C_i\right) = P\left(x_1^t, \cdots, x_d^t \middle| C_i\right) = \prod_{j=1}^{d} P\left(x_j^t \middle| C_i\right) = \prod_{j=1}^{d} \prod_{k=1}^{n_j} P_{ijk}^{z^t_{jk}}$$

- Discriimant function

$$g_i\left(\boldsymbol{x}^t\right) = \log P\left(\boldsymbol{x}^t \middle| C_i\right) + \log P\left(C_i\right) = \sum_{j=1}^{d} \sum_{k=1}^{n_j} z^t_{jk} \log P_{ijk} + \log P\left(C_i\right)$$

  - Linear ?

- Maximum likelihood estimation for $P_{ijk}$

$$\hat{P}_{ijk} = \frac{\sum_{t=1}^{N} z^t_{jk} \cdot r_i^t}{\sum_{t=1}^{N} r_i^t}$$