Constraint Satisfaction Problems

Berlin Chen 2005

References:

1. S. Russell and P. Norvig. *Artificial Intelligence: A Modern Approach*. Chapter 5
2. S. Russell's teaching materials

Introduction

- Standard Search Problems
  - State is a "black box" with no discernible internal structure
  - Accessed by the goal test function, heuristic function, successor function, etc.

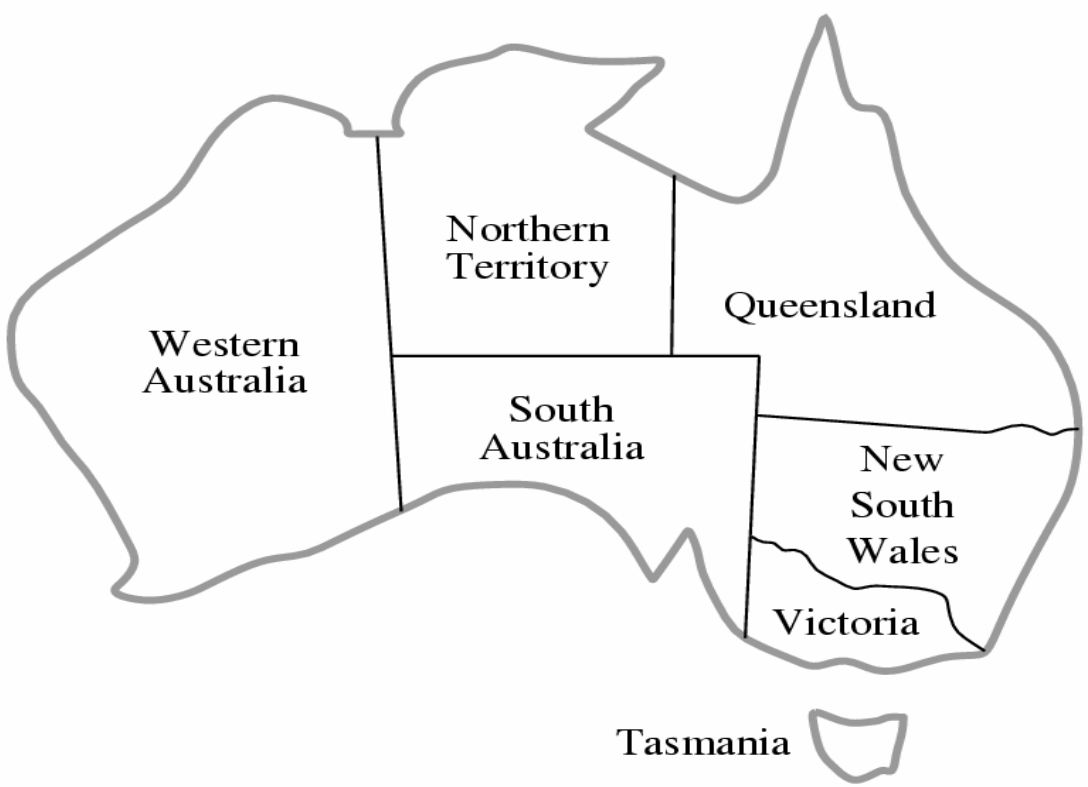
- Constraint Satisfaction Problems (CSPs)
  - State and goal test conform to a standard, structured, and very simple representation
    - State is defined by variables X_i with values v_i from domain D_i
    - Goal test is a set of constraints $C_1, C_2, .., C_m$, which specifies allowable combinations of values for subsets of variables
  - Some CSPs require a solution that maximizes an objective function

Derive heuristics without domain-specific knowledge

Introduction (cont.)

- Consistency and completeness of a CSP
  - Consistent (or called *legal*)
    - An assignment that does not violate any constraints
  - Complete
    - Every variable is assigned with a value
    - A **"solution"** to a CSP is a complete assignment satisfying all the constraints

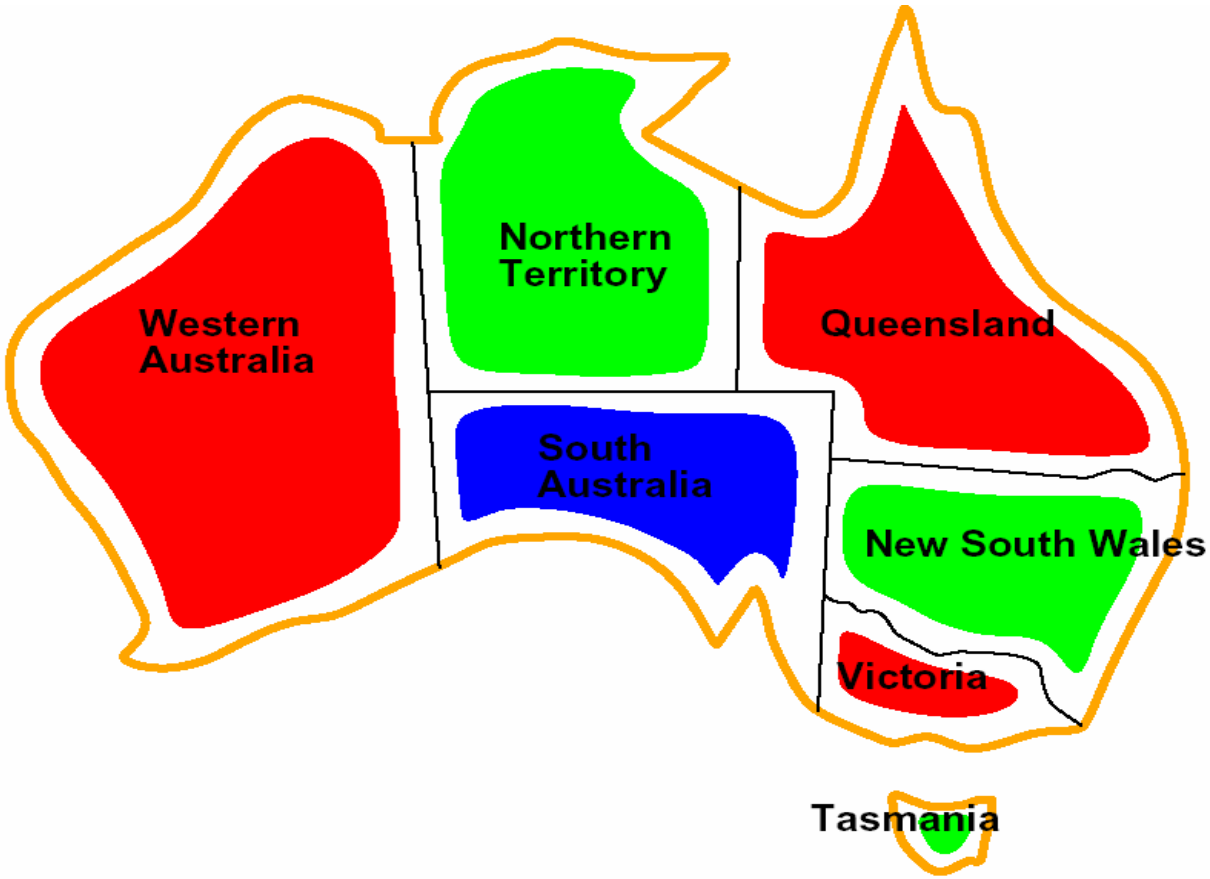
Example: Map-Coloring Problem



- Variables: *WA*, *NT*, *Q*, *NSW, V, SA, T*
- Domains: D_i= {*red*, *green*, *blue*}
- Constraints: neighboring regions must have different colors

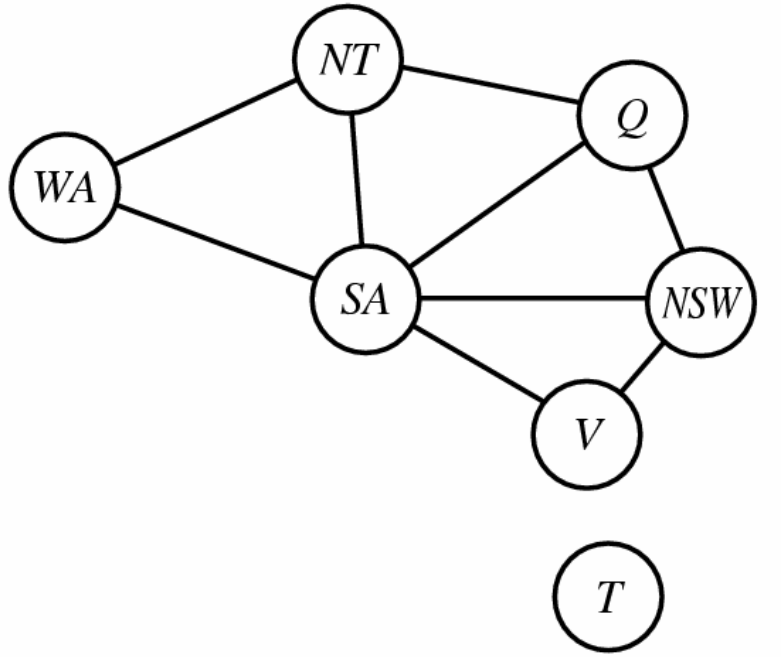
Example: Map-Coloring Problem (cont.)

- Solutions: assignments satisfying all constraints, e.g.,
  {*WA=red*, *NT=green*, *Q=red*, *NSW=green*, *V=red*, *SA=blue*, *T=green*}



Example: Map-Coloring Problem (cont.)

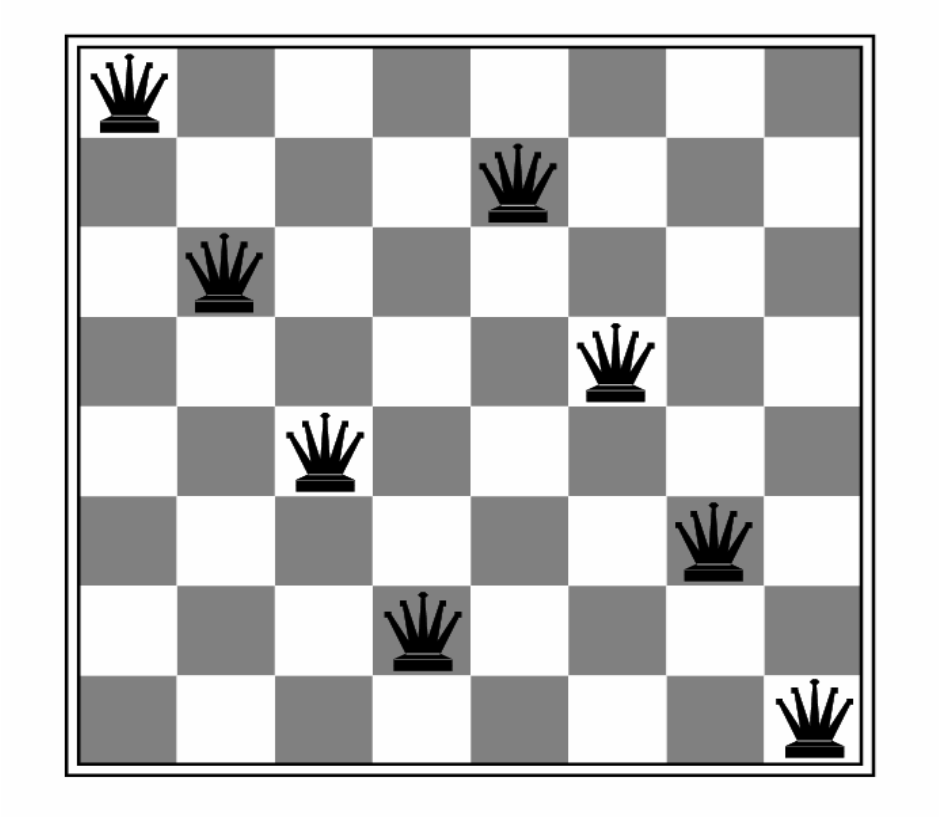
- The CSP can be visualized as a **Constraint Graph**
  - Nodes: correspond to *variables*
  - Arcs: correspond to *constraints*



Constraint Graph

  - A visualization of representation of (binary) constraints

Example: 8-Queens Problem



- Variables: Q_1, Q_2,…, Q_8
- Domains: D_i= {1, 2, …, 8}
- Constraints: no queens at the same row, column, and diagonal

Benefits of CSPs

- Conform the problem representation to a standard pattern
  - A set of variables with assigned values
- Generic heuristics can be developed with no domain-specific expertise
- The structure of constraint graph can be used to simplify the solution process
  - Exponential reduction

Formulation

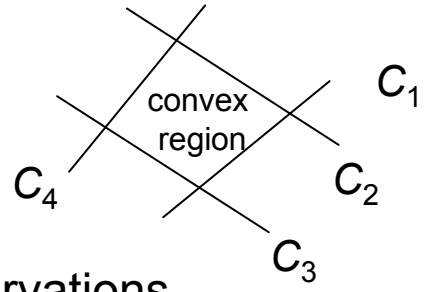
- Incremental formulation
  - **Initial state**: empty assignment { }
  - **Successor function**: a value can be assigned to any unassigned variables, provided that no conflict occurs
  - **Goal test**: the assignment is complete
  - **Path cost**: a constant for each step

- Complete formulation
  - Every state is a complete assignment that may or may not satisfies the constraints
  - Local search can be applied

CSPs can be formulated as search problems

Variables and Domains

- Discrete variables
  - Finite domains (size d)
    - E.g., color-mapping (d colors), Boolean CSPs (variables are either true or false, d=2), etc.
    - Number of complete assignment: $O(d^n)$ (exponential in number of variables)
  - Infinite domains (integers, strings, etc.)
    - Job scheduling, variables are start and end days for each job
    - A constraint language is needed, e.g., $StartJob_1 + 5 \leq StartJob_3$
      - Linear constraints are solvable, while nonlinear constraints un-decidable
    - How to convert to a finite-domain problem ?

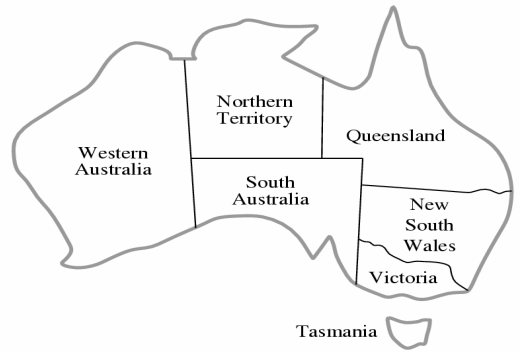


- Continuous variables
  - E.g., start and end times for Hubble Telescope observations
  - Linear constraints are solvable in polynomial time by **linear programming** methods

Constraints

absolute constraints

- Unary constraints
  - Restrict the value of a single variable
  - E.g., $SA \neq green$
  - Can be simply preprocessed before search
- Binary constraints
  - Restrict the values of a pair of variables
  - E.g., $SA \neq WA$
  - Can be represented as a constraint graph
- High-order constraints
  - Three or more variables are involved when the value-assigning constriction is considered
  - E.g., column constraints in the cryptarithmetic problem
- Preference (soft) constraints
  - A cost for each variable assignment
  - E.g., the university timetabling problem
  - Can be viewed as constrained optimization problems

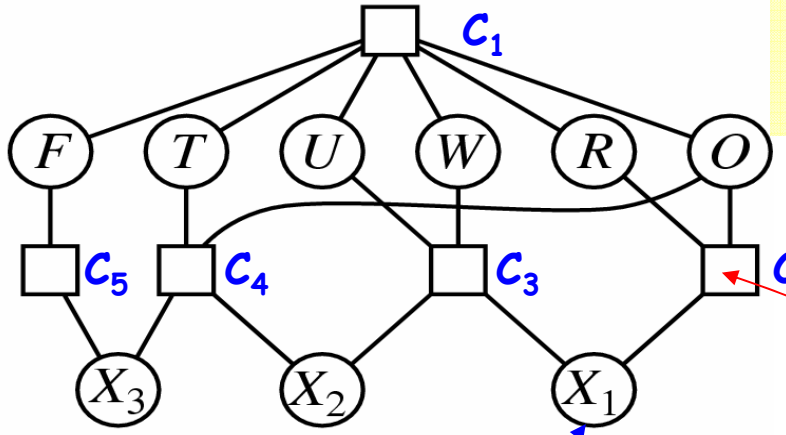


Constraints (cont.)

- Example: the cryptarithmetic problem (high-order constraints)

$$\begin{array}{cccc} & T & W & O \\ + & T & W & O \\ \hline F & O & U & R \end{array}$$

(a)



(b)

constraint hypergraph

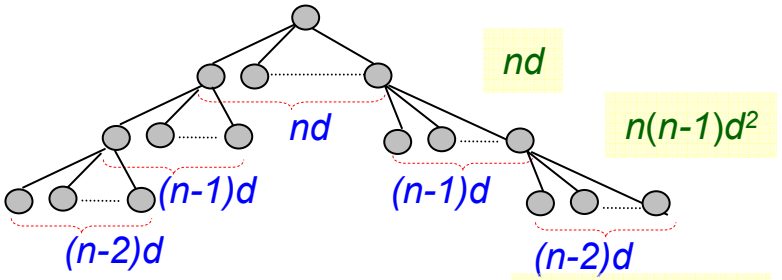
constraint

auxiliary variable

- Variables: F, T, U, W, R, O, X_1, X_2, X_3
- Domains: {0,1, 2, …, 9} and {0,1}
- Constraints:
  - *Alldiff* (F, T, U, W, R, O) C_1
  - $O+O=R+10\cdot X_1$ C_2
  - $X_1+W+W=U+10\cdot X_2$ C_3
  - $X_2+T+T=O+10\cdot X_3$ C_4
  - $X_3=F$ C_5

Standard Search Approach

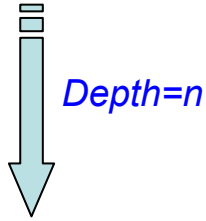
- If **incremental formulation** is used
- Breadth-first search with search tree with depth limit *n*



Initial state: empty assignment {}

Successor function: a value can be assigned to any unassigned variables, provided that no conflict occurs

Goal test: the assignment is complete



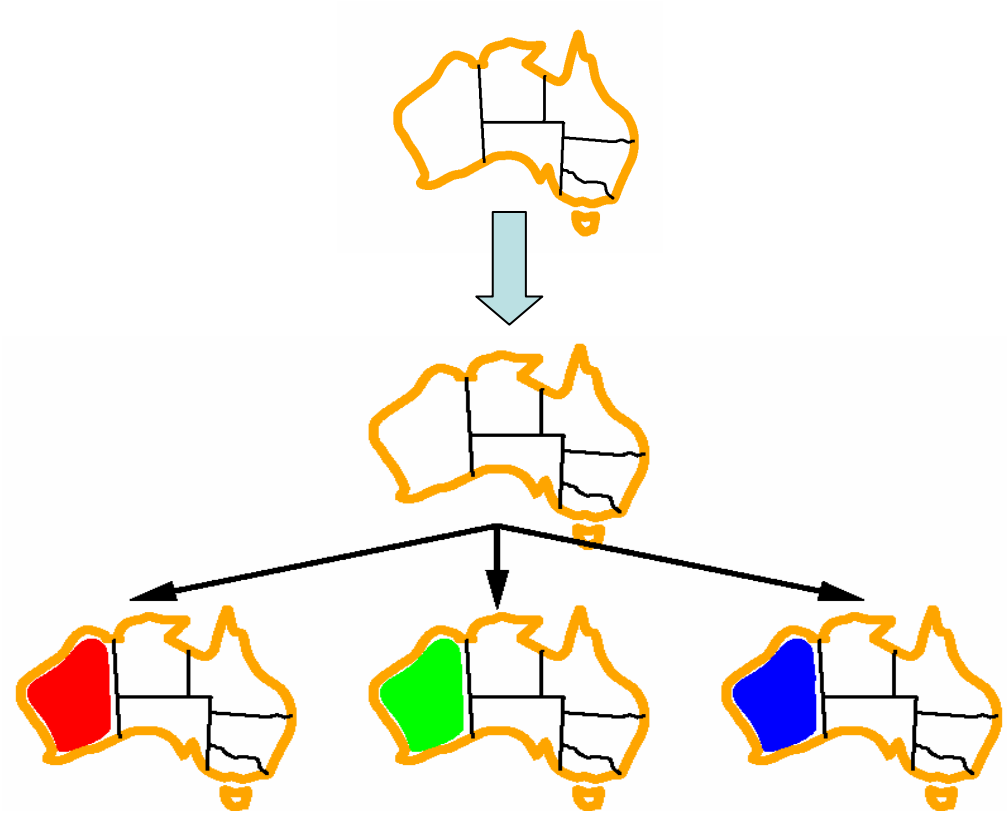
Variables (*n*) and Values (*d*)

$n!d^n$ → Totally, d^n distinct leaf nodes (because of commutativity)

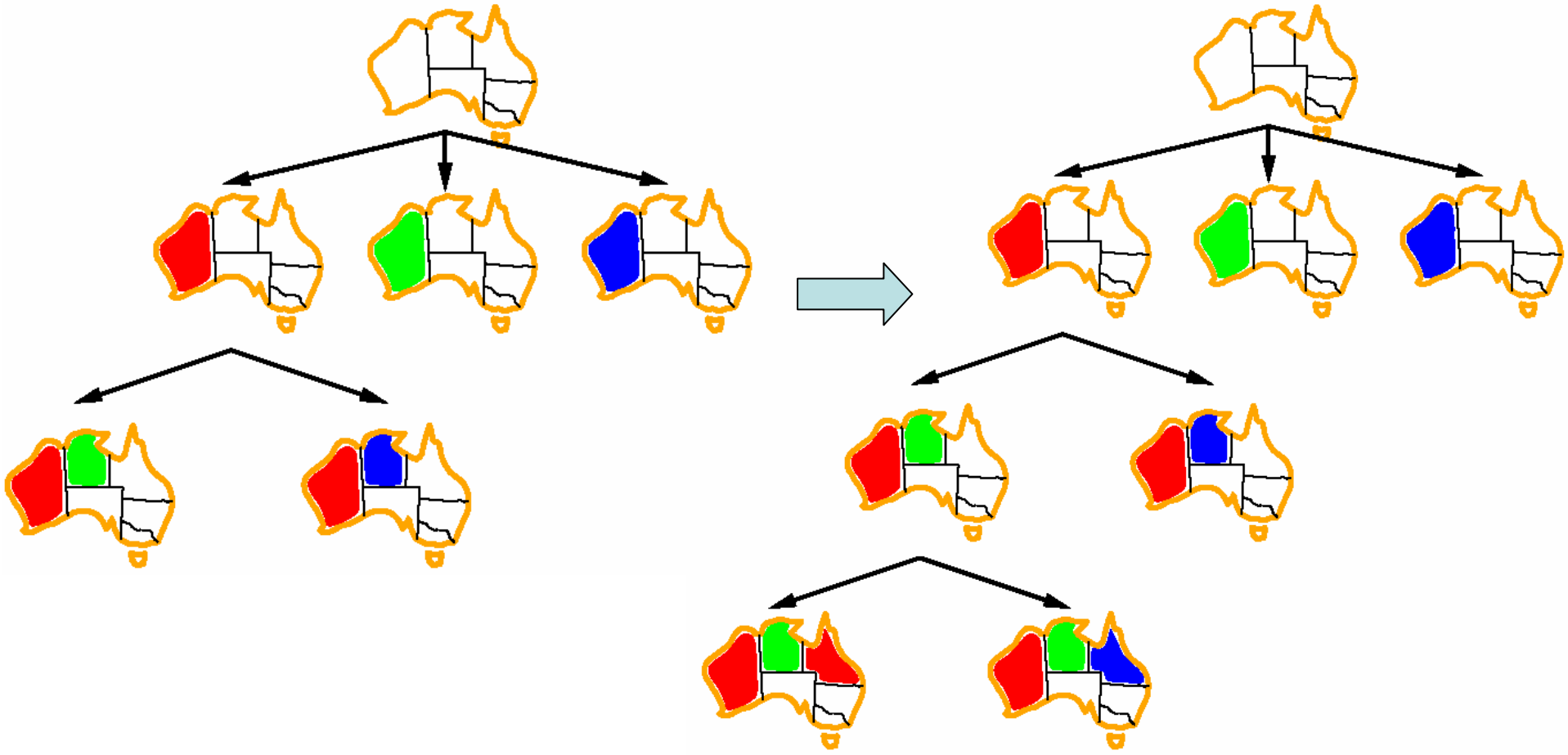
  - Every solution appears at depth *n* with *n* variable assigned
  - DFS (or depth-limited search) also can be applied (smaller space requirement)
  - The order of assignment is not important

Backtracking Search

- **DFS** for CSPs (uninformed search)
  - One variable is considered orderly at a time (level) for expansion
  - Backtrack when no legal values left to assign
- The basic uniformed search for CSPs



Backtracking Search (cont.)



Backtracking Search (cont.)

- Algorithm

```
function BACKTRACKING-SEARCH(csp) returns a solution, or failure
   return RECURSIVE-BACKTRACKING({}, csp)

function RECURSIVE-BACKTRACKING(assignment, csp) returns a solution, or failure
   if assignment is complete then return assignment
   var ← SELECT-UNASSIGNED-VARIABLE(VARIABLES[csp], assignment, csp)
   for each value in ORDER-DOMAIN-VALUES(var, assignment, csp) do
      if value is consistent with assignment according to CONSTRAINTS[csp] then
         add {var = value} to assignment
         result ← RECURSIVE-BACKTRACKING(assignment, csp)
         if result ≠ failure then return result
         remove {var = value} from assignment
   return failure
```

list of unassigned variables

decide which variable

decide which value

  - When it fails: back up to the preceding variable and try a different value of it
    - Chronological backtracking

Improving Backtracking Efficiency

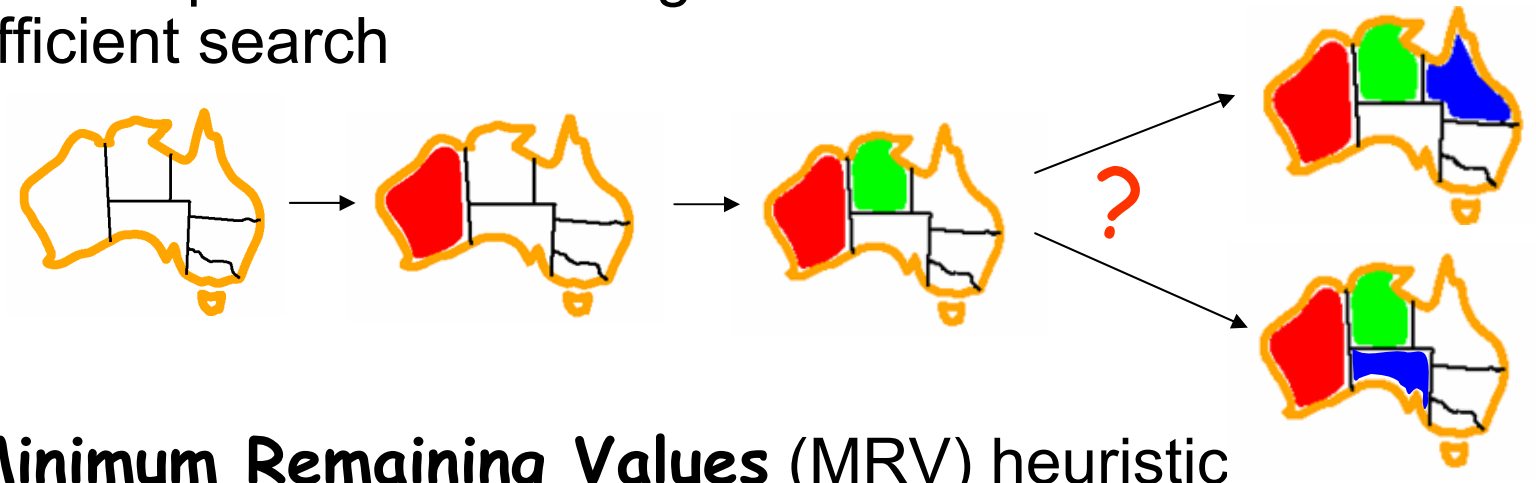
- General-purpose methods help to speedup the search
  - What variable should be considered next?
  - In what order should variable's values be tried?
  - Can we detect the inevitable failure early?
  - Can we take advantage of problem structure?

Improving Backtracking Efficiency (cont.)

- Variable Order
  - Minimum remaining value (MRV)
    - Also called ”most constrained variable”, “fail-first”
  - Degree heuristic
    - Act as a “tie-breaker”
- Value Order
  - Least constraining value
    - If full search, value order does not matter
- Propagate information through constraints
  - Forward checking
  - Constraint propagation

Variable Ordering

- The simple static ordering seldom results in the most efficient search

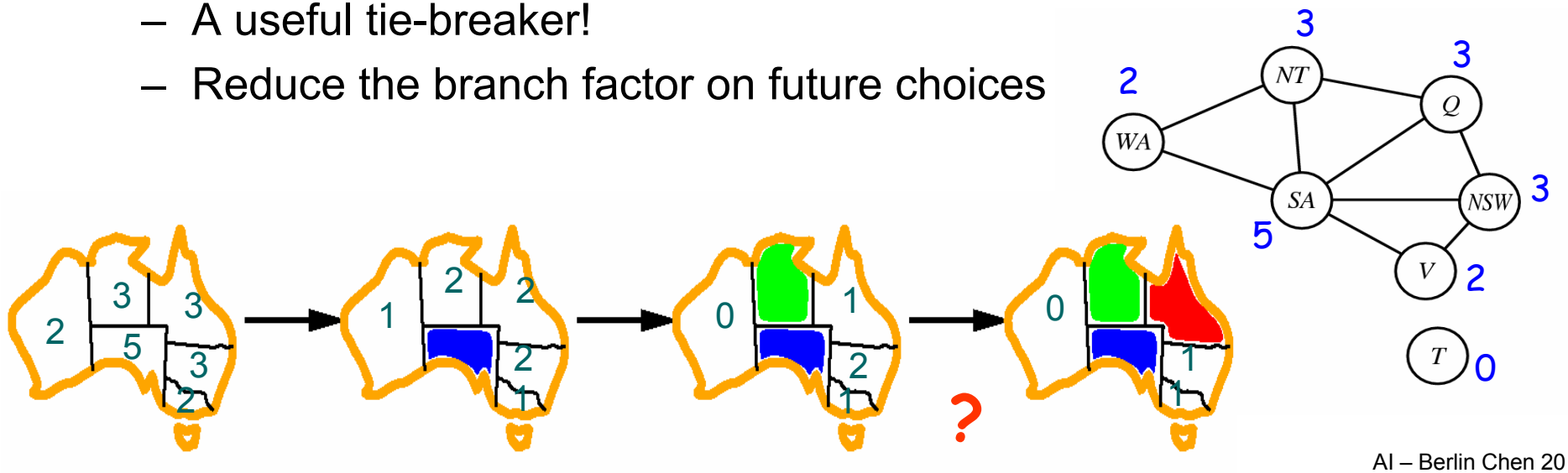


- **Minimum Remaining Values** (MRV) heuristic
  - Also called “most constrained variable” or “fail-first” heuristic
  - Choose the **variable** with the most constraints (on values) from the remaining variables
    - If a variable *X* with zero legal values remained, MRV selects it and causes a failure immediately
    - The search tree can be therefore pruned
      - Reduce the number of branch factor at lower levels ASAP

Variable Ordering (cont.)

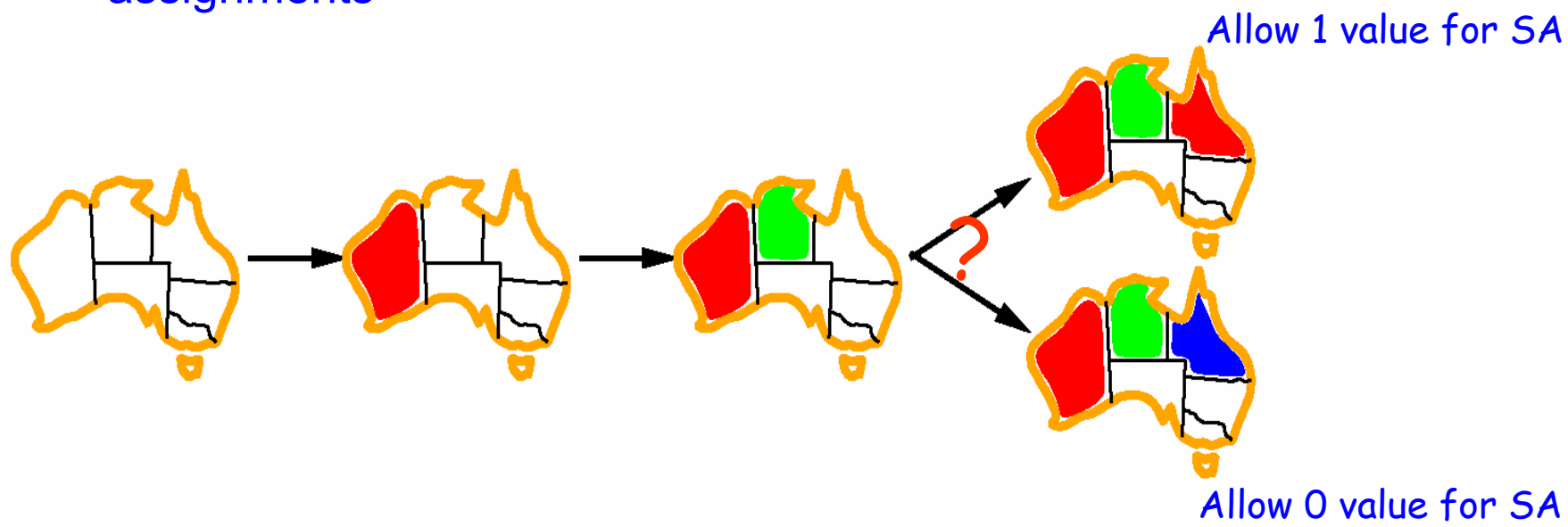
- MRV doesn't help at all in choosing the first region to color in Australia
  - All regions have three legal colors

- So, the **degree heuristic** can be further applied
  - Select the **variable** that is involved in the largest number of constraints on other unassigned variables
  - A useful tie-breaker!
  - Reduce the branch factor on future choices



Value Ordering

- **Least-Constraining-Value** heuristic
  - Given a variable, choose the value that rules out the fewest chooses of values for the remaining (neighboring) variables
  - I.e., leave the maximum flexibility for subsequent variable assignments



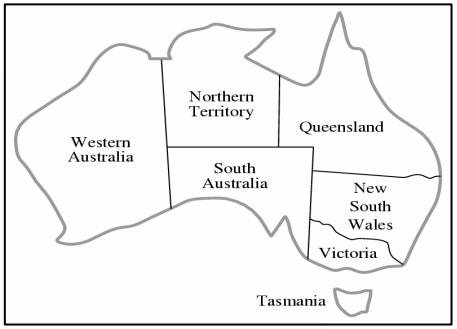
- If all the solutions (not just the first one) are needed, the value ordering doesn't matter

Problem	Backtracking	BT+MRV	Forward Checking	FC+MRV	Min-Conflicts
USA	(> 1,000K)	(> 1,000K)	2K	60	64
n-Queens	(> 40,000K)	13,500K	(> 40,000K)	817K	4K
Zebra	3,859K	1K	35K	0.5K	2K
Random 1	415K	3K	26K	2K	
Random 2	942K	27K	77K	15K	

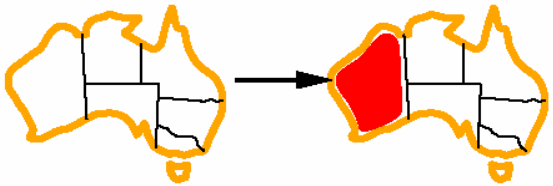
Figure 5.5 Comparison of various CSP algorithms on various problems. The algorithms from left to right, are simple backtracking, backtracking with the MRV heuristic, forward checking, forward checking with MRV, and minimum conflicts local search. Listed in each cell is the median number of consistency checks (over five runs) required to solve the problem; note that all entries except the two in the upper right are in thousands (K). Numbers in parentheses mean that no answer was found in the allotted number of checks. The first problem is finding a 4-coloring for the 50 states of the United States of America. The remaining problems are taken from Bacchus and van Run (1995), Table 1. The second problem counts the total number of checks required to solve all n-Queens problems for n from 2 to 50. The third is the "Zebra Puzzle," as described in Exercise 5.13. The last two are artificial random problems. (Min-conflicts was not run on these.) The results suggest that forward checking with the MRV heuristic is better on all these problems than the other backtracking algorithms, but not always better than min-conflicts local search.

Forward Checking

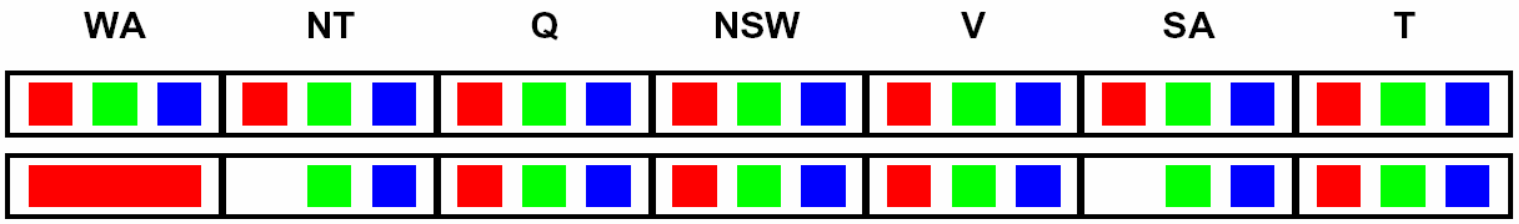
- Not only consider constraints on a variable!
- But propagate constraint information from assigned variables to connected unassigned variables
- Keep track of remaining legal values for unsigned variables, and terminate the search when any variable has no legal values
  - Remove the inconsistent value of the unassigned variable
  - Before searching is performed on the unsigned variables



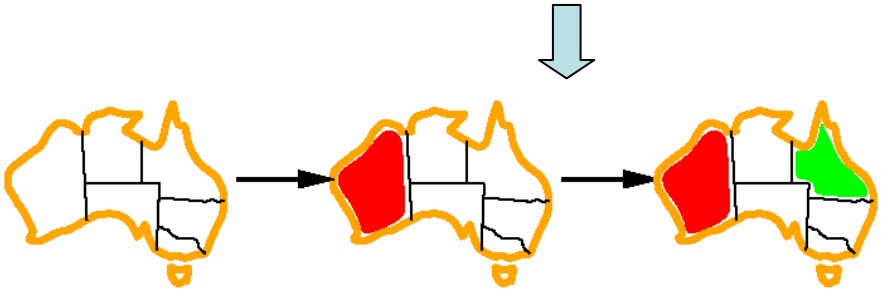
Forward Checking (cont.)

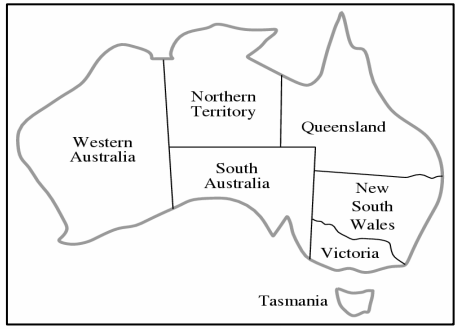


Note: MRV, degree heuristic etc., were not used here



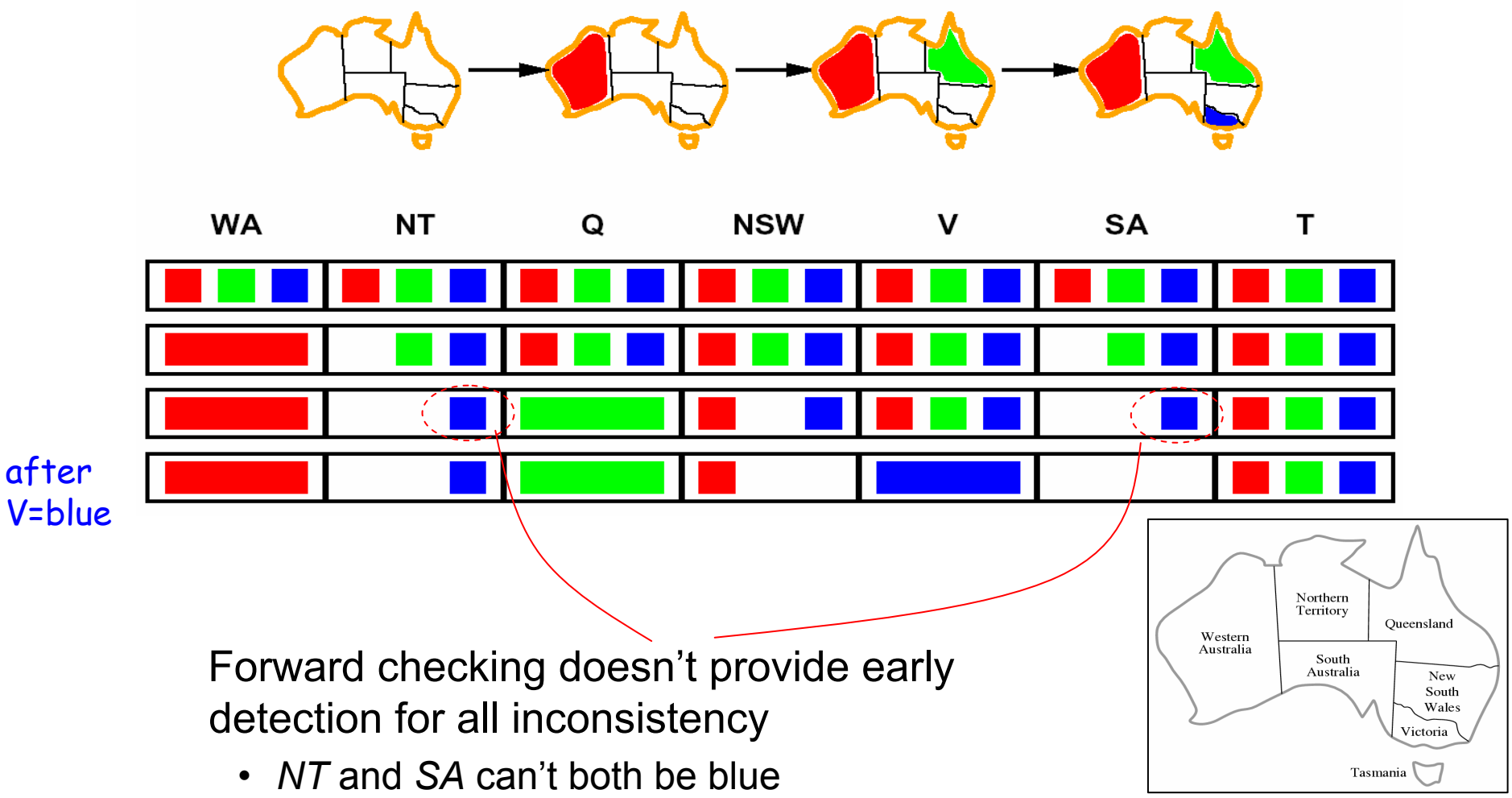
after WA=red





after Q=green

Forward Checking (cont.)



Forward checking doesn't provide early detection for all inconsistency

- *NT* and *SA* can't both be blue

Constraint Propagation

- Repeated enforce constraints locally
- Propagate the implications of a constraint on one variable onto other variables
- Method
  - Arc consistency

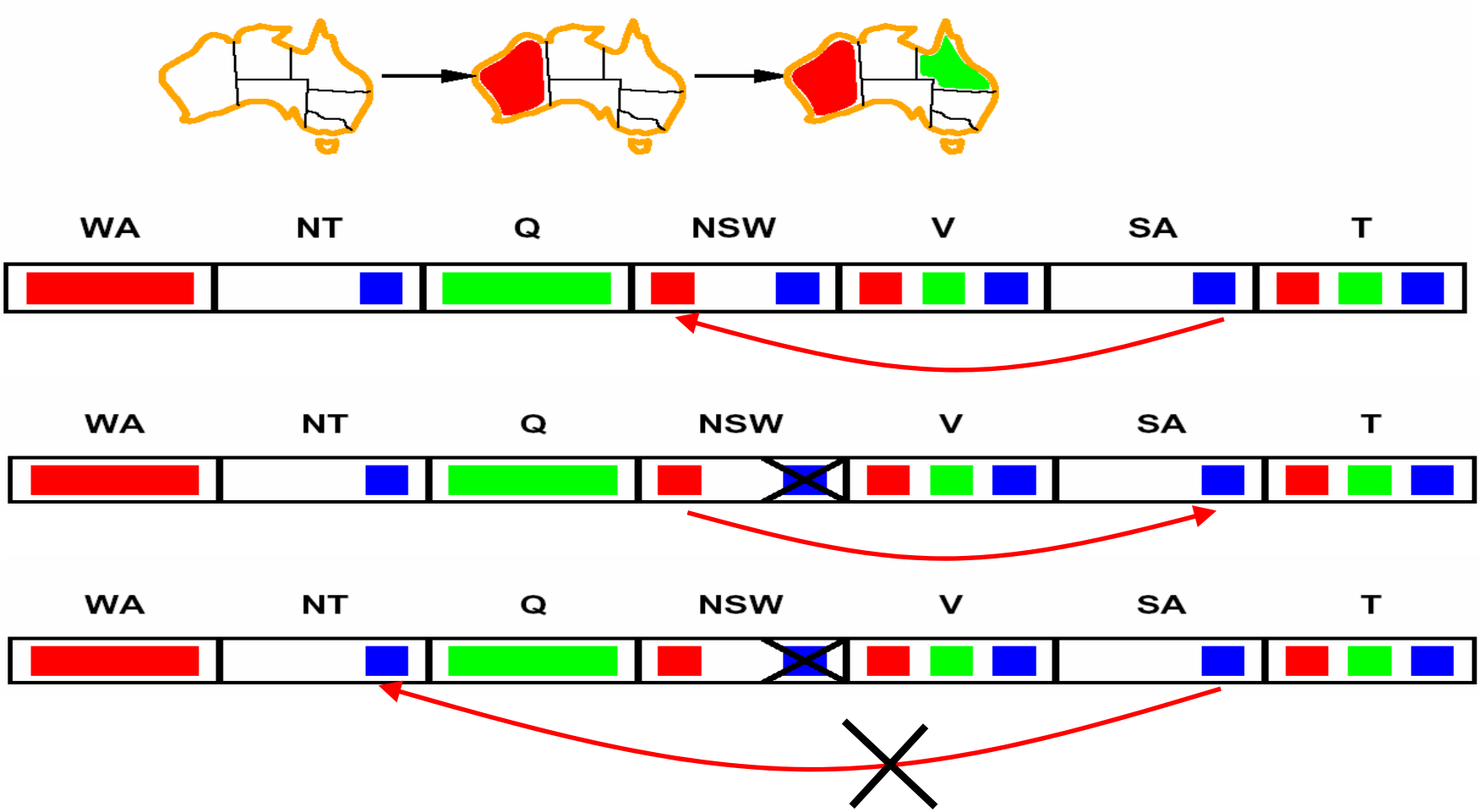
Arc Consistency

- $X \rightarrow Y$ is consistent iff

  for every value *x* of *X* there is some value *y* of *Y* that is consistent (allowable)

  – A method for implementing constraint propagation exists
  – Substantially stronger than forward checking

Arc Consistency (cont.)



- If *X* loses a value, neighbors of *X* need to be rechecked
- Arc consistency detects failure earlier than forward checking
- Can be run as a preprocessor or after each assignment

Arc Consistency (cont.)

- Algorithm $O(n^2 d^3)$

function AC-3(*csp*) **returns** the CSP, possibly with reduced domains
  inputs: *csp*, a binary CSP with variables $\{X_1, X_2, \ldots, X_n\}$
  local variables: *queue*, a queue of arcs, initially all the arcs in *csp* $O(n^2)$ $C_2^n = \frac{n \times (n-1)}{2}$

  while *queue* is not empty **do**
    $(X_i, X_j) \leftarrow$ REMOVE-FIRST(*queue*)
    if REMOVE-INCONSISTENT-VALUES(X_i, X_j) **then**
      for each X_k **in** NEIGHBORS[X_i] **do**
        add (X_k, X_i) to *queue*

If some values of a nodes X_i is removed, arcs pointing to it (X_k, X_i) must be reinserted on the queue for checking again

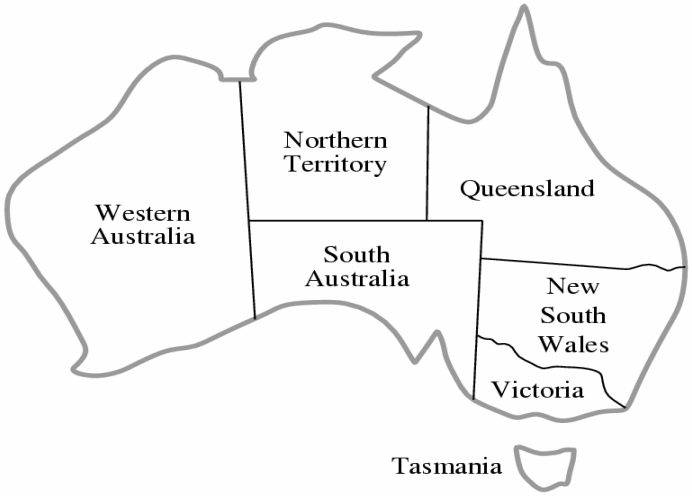
$O(d)$ each variable at most has d values to be removed

function REMOVE-INCONSISTENT-VALUES(X_i, X_j) **returns** true iff we remove a value
  removed $\leftarrow$ *false*
  for each x **in** DOMAIN[X_i] **do**
    if no value y in DOMAIN[X_j] allows (x,y) to satisfy the constraint between X_i and X_j
      then delete x from DOMAIN[X_i]; *removed* $\leftarrow$ *true* $O(d^2)$
  return *removed*

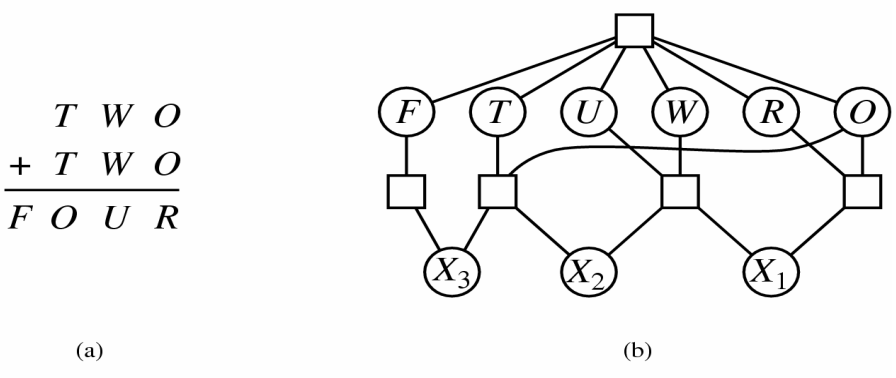
d: number of values in the domain of each variable

Arc Consistency (cont.)

- Arc consistency doesn't reveal every possible inconsistency !
  - E.g. a particular assignment {*WA*=*red*, *NSW*=*red*} which is inconsistent but can't be found by arc consistency algorithm
    - *NT*,*SA*,*Q* have two colors left for assignments
  - Arc consistency is just 2-consistency
    - 1-consistency, 2-consistency,…, k-consistency, etc.

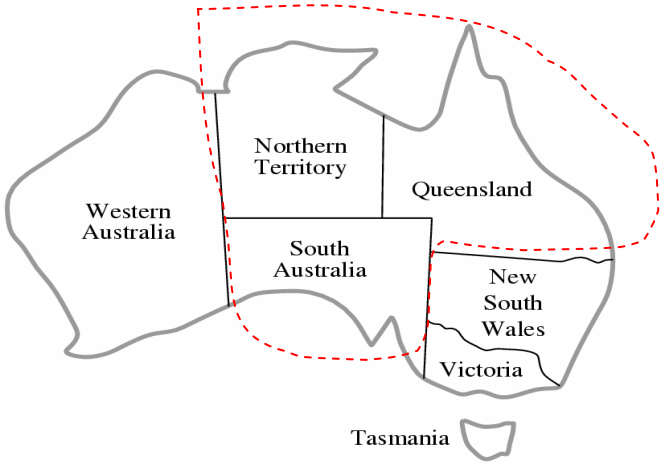


Handling Special Constraints



Alldiff (*F*, *T*, *U*, *W* , *R*, *O*)
variables *m*
value *n*
m>*n* ?

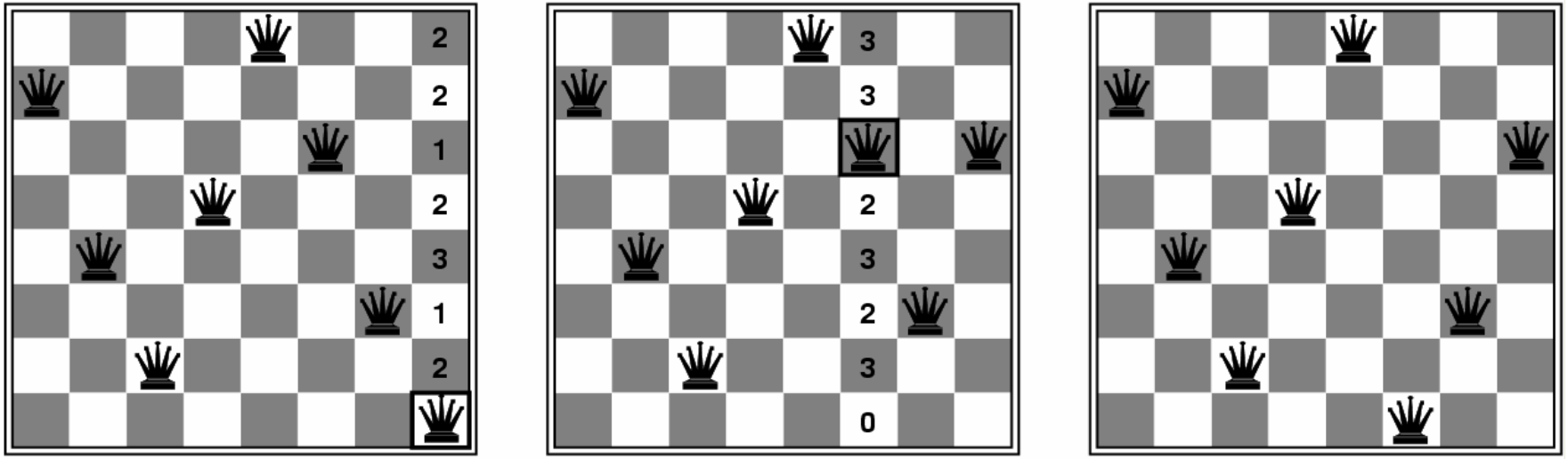
Alldiff (*NT*, *SA*,*Q*)
variables m=*3*
value n= *2*
m>*n* ?



Local Search for CSPs

- If complete formulation is used
- Local search can easily be extended to CSPs with objection functions
  - Hill-climbing, simulated annealing etc. can be applied
- Method
  - Allow states with unsatisfied constraints
  - Operators
    - reassign variable values
  - Variable selection
    - Randomly select any conflict variable
  - E.g. the “min-conflicts” heuristic
    - For a given variable, selecting the value that results the minimum number of conflicts with other variables
    - E.g., hill-climbing with $h(n)$=total number of violated constraints

Local Search for CSPs (cont.)



- Especially suitable for problems for on-line settings

Local Search for CSPs (cont.)

```
function MIN-CONFLICTS(csp, max_steps) returns a solution or failure
  inputs: csp, a constraint satisfaction problem
          max_steps, the number of steps allowed before giving up
  current ← an initial complete assignment for csp
  for i = 1 to max_steps do
      if current is a solution for csp then return current
      var ← a randomly chosen, conflicted variable from VARIABLES[csp]
      value ← the value v for var that minimizes CONFLICTS(var, v, current, csp)
      set var = value in current
  return failure
```

initialization (randomly generated or …)

randomly select a variable

select the value of the variable with minimum conflicts

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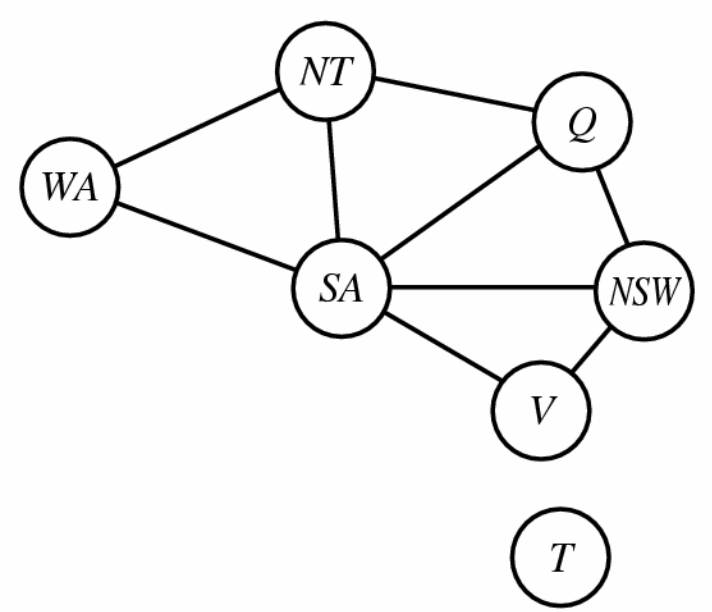
Figure 5.5 Comparison of various CSP algorithms on various problems. The algorithms from left to right, are simple backtracking, backtracking with the MRV heuristic, forward checking, forward checking with MRV, and minimum conflicts local search. Listed in each cell is the median number of consistency checks (over five runs) required to solve the problem; note that all entries except the two in the upper right are in thousands (K). Numbers in parentheses mean that no answer was found in the allotted number of checks. The first problem is finding a 4-coloring for the 50 states of the United States of America. The remaining problems are taken from Bacchus and van Run (1995), Table 1. The second problem counts the total number of checks required to solve all n-Queens problems for n from 2 to 50. The third is the "Zebra Puzzle," as described in Exercise 5.13. The last two are artificial random problems. (Min-conflicts was not run on these.) The results suggest that forward checking with the MRV heuristic is better on all these problems than the other backtracking algorithms, but not always better than min-conflicts local search.

Problem Structure

- The structure of the problem represented by the constraint graph can be used to find solutions quickly



  - E.g., Tasmania and the mainland are independent sub-problems

- Identify the connected components (as sub-problems) of constraint graph to reduce the solution time

Problem Structure (cont.)

- Suppose that each sub-problem has *c* variables out of *n* total
  - With decomposition
    - Worse-case solution cost: $n/c \cdot d^c$ linear in *n*
  - Without decomposition
    - Worse-case solution cost: d^n exponential in *n*
  - E.g., *n*=80, *d*=2, *c*=20

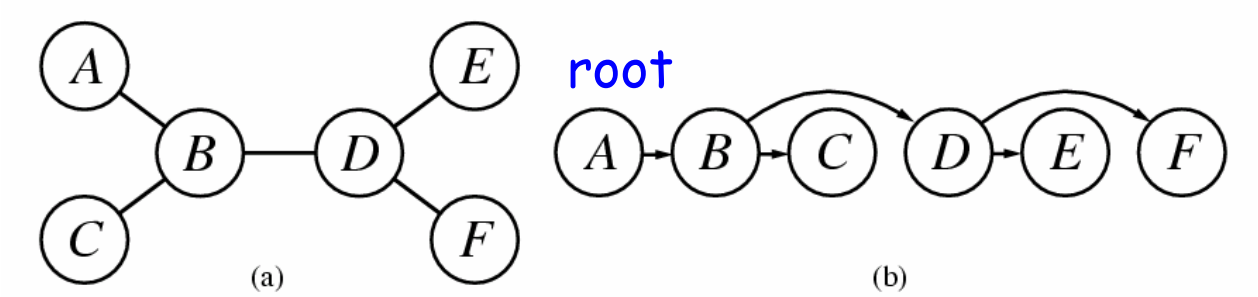
    2^{80}=40 billion year at 10 million nodes/sec

    $4x2^{20}$=0.4 seconds at 10 million nodes/sec

- Completely independent sub-problems are rare
  - Sub-problems of a CSP are often connected

Tree-Structured CSPs

- Tree-Structured CSPs are the simplest ones
  - Can be solved in time linear in the number of variables

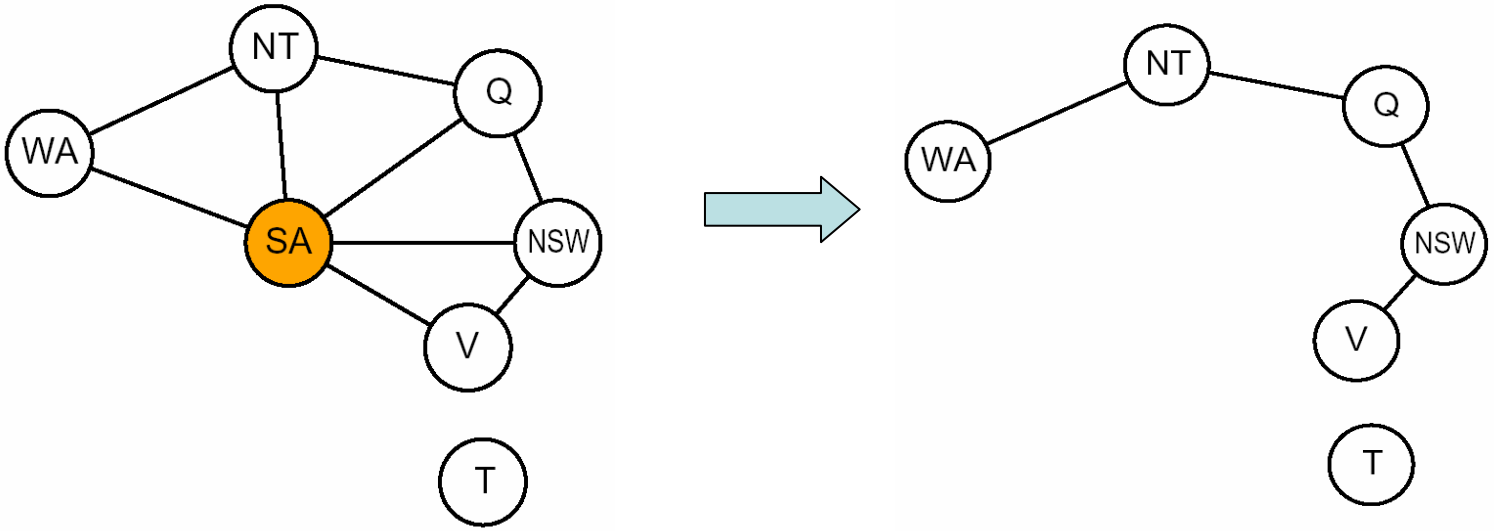


- Algorithm for tree-structured CSPs
  1. Choose a variable as the root, order variables from root to leaves such that every node's parent precedes it in the ordering
  2. For j from n down to 2, apply arc consistency to the arc (X_i, X_j), where X_i is the parent of X_j, remove the values from Domain[X_i] as necessary $O(nd^2)$, if no loops
  3. For j from 1 to n, assign X_j consistently with parent X_i

Why in reverse order ?

Reducing Constraint Graphs to Trees

- Method 1 $O(d^c(n-c)d^2)$
  - Initiate a subset of variables S (cycle cutset, with size c) such that the remaining constraint graph is a tree
  - Prune the domains of the remaining variables that are inconsistent with S $O((n-c)d^2)$ ← Tree-structure CSP
  - If the remaining CSP has a solution, return it together with the assignment for S $O(d^c)$



  - E.g., if the value assigned to SA is a wrong one, do again !

Reducing Constraint Graphs to Trees (cont.)

- Method 2
  - Construct a tree decomposition of the constraint graph into a set of connected subproblems
  - Each subproblem is solved independently and the resulting solutions are then combined
  - Properties
    - Every variable must appear in at least one subproblem
    - Two variables connected by a constraint must appear together
    - A variable connecting some subproblems must appear in all of them